CURVES OF PERIOD TWO POINTS FOR TRACE MAPS

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Abstract. We consider an infinite family of trace maps α_n and their action on $\mathbb{R}^3$. Trace maps fix certain invariant surfaces, and in an earlier paper we found that the fixed points for α_n on one such surface were joined in pairs by curves of fixed points, thus determining a 'duality' for such fixed points. We now extend this idea to determine the duality for all the points of period 2 that lie in the planes $x = \pm y$ and for certain others that do not.

Keywords: Trace map, periodic point, curve of fixed points, Chebyshev polynomial, area preserving map.

2010 AMS Classification: Primary 37C25 37D40 37E99; Secondary 42C05

§1 Introduction

We are studying a family α_n of maps from a 3-ball $\mathcal{T} \subset \mathbb{R}^3$ to itself and how a disc $\mathcal{P}$ in a plane of symmetry of $\mathcal{T}$ meets its image $(\mathcal{P})\alpha_n$. In [HM] we found the pattern of curves of fixed points of α_n and Figure 2 shows those in $\mathcal{P}$. A certain toral automorphism reveals how $(\partial\mathcal{P})\alpha_n$ winds back and forth in $\partial\mathcal{T}$, and, in this paper, we discover how $\partial\mathcal{P} \cap (\partial\mathcal{P})\alpha_n$ determines the curves of period 2 points in $\mathcal{P}$, some of which are shown in Figure 1.

Let us now describe the situation more formally. Given $A, B \in SL(2, \mathbb{R})$, the map $(A, B) \mapsto (\mathrm{trace}(A), \mathrm{trace}(B), \mathrm{trace}(AB))$ can be used to associate a polynomial diffeomorphism of $\mathbb{R}^3$ to each automorphism of the (non-abelian) free group of rank 2. These diffeomorphisms are called *trace maps* and have been widely studied, see [Cas, Can, RB1, RB2, ABG, Ig, BR, LW, PWW]. Each trace map gives a 1-parameter family of area preserving maps of certain level surfaces that foliate $\mathbb{R}^3$. The restriction to one special surface $\partial\mathcal{T}$ (where $\mathcal{T}$ is a 3-ball) is covered by a well-understood linear action on the 2-torus, and the set of periodic points is dense there. In [HM] we defined a family of trace maps and determined their curves of fixed points. For each trace map α_n in our family, we determined pairs of fixed points in the surface $\partial\mathcal{T}$ that are dual in the sense of being connected by a curve of fixed points of α_n. This gave a *duality* for all fixed points of α_n on $\partial\mathcal{T}$.

In this paper we shall study, for α_n in the same family of trace maps, all those points of period 2 that lie in the planes of symmetry $x = \pm y$ of $\mathbb{R}^3$ (in §7), and then some of the points of period 2 that are not in these planes (in §8). In Theorem 7.5 we determine the duality of points in the plane $x = y$ (or $x = -y$) and the pattern of the curves of points of period 2 joining them (which, according to §5, also lie in that plane, so we are studying the intersection of that plane and its image). This will involve a study of multiplication modulo 1 by $n+1$ and by $n-1$ in the interval $[0, 1]$ partitioned into $2n$ equal subintervals. In Theorem 8.13 we look at period 2 points not in these planes; this involves computations with Chebyshev polynomials.

As can be seen in Figure 1, the curves of period 2 points in the $x = \pm y$ planes mostly lie in corridors separated by vertical lines; one curve that crosses such a vertical line forms with that line a lower triangle that encloses curves that we shall call *stalagmites* and an upper triangle that encloses curves that we shall call *stalactites*, while other curves not thus enclosed reach from top to bottom (and we shall call them *columns*). It is this pattern and the duality that it determines that we shall explore in §§2-7 of this paper. We introduce our family of trace maps in §2 and their curves of *fixed* points in §3, see Figure 2, while the properties of Chebyshev polynomials needed are collected in §4. General properties of the curves of *period two* points in the plane $x = y$ are developed in §5 and the curves that are *symmetric* (in the sense that they are symmetric to their image, see Figure 3) are studied in §6. In §7 we study which corridor contains the image of each period two point from $\partial\mathcal{T}$ in a given corridor to determine which pairs of these points are dual. Finally, in §8 we study various curves of period 2 and 4 points that are given by equations involving Chebyshev polynomials of the second kind. Duality is determined for such curves also.

§2 Preliminaries

In this section we will describe general trace maps, some of their properties, and the family of trace maps that we study.

Let $F_2 = \langle x_1, x_2\rangle$ be a free group of rank 2 and let $\sigma_i \in \mathrm{Aut}(F_2), i = 1, 2$, be defined by

$$\sigma_1(x_1) = x_1x_2, \quad \sigma_1(x_2) = x_2;$$
$$\sigma_2(x_1) = x_1, \quad \sigma_2(x_2) = x_1^{-1}x_2.$$

One can show that σ_1, σ_2 satisfy the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. We thus have a representation of the braid group B_3 [Bi]. Note [MKS, Theorem 3.9] that any element of $\mathrm{Aut}(F_2)$ fixes the commutator $x_1x_2x_1^{-1}x_2^{-1}$ (up to conjugacy).

Now suppose that the $x_i, i = 1, 2$, are represented by elements of $SL(2, \mathbb{C})$, that we also denote by x_i. Let

$$x = \mathrm{trace}(x_1)/2, \quad y = \mathrm{trace}(x_2)/2, \quad z = \mathrm{trace}(x_1x_2)/2.$$

Recall the standard trace identities for such 2×2 matrices:

$$\mathrm{trace}(A^{-1}) = \mathrm{trace}(A), \quad \mathrm{trace}(I_2) = 2,$$
$$\mathrm{trace}(AB) = \mathrm{trace}(A)\mathrm{trace}(B) - \mathrm{trace}(AB^{-1}).$$

Using these relations one can prove the well-known fact that if w is a word in $x_1, x_2, x_1^{-1}, x_2^{-1}$, then $\mathrm{trace}(w)$ is an integer polynomial in x, y, z. Thus we obtain the following induced action of σ_1, σ_2 on $\mathbb{Q}[x, y, z]$:

$$\begin{aligned}\sigma_1(x, y, z) &= (z, y, 2yz - x)\\ \sigma_2(x, y, z) &= (x, 2xy - z, y).\end{aligned} \tag{2.1}$$

Now because this action is obtained using the action on traces one should expect that this only guarantees an action of B_3 if we consider the action on the *trace ring* [Ma], this being the quotient of $\mathbb{Q}[x, y, z]$ by all generic trace relations. In terms of the generators x, y, z this is the quotient of $\mathbb{Q}[x, y, z]$ by the ideal generated by the element $E - 1$ where

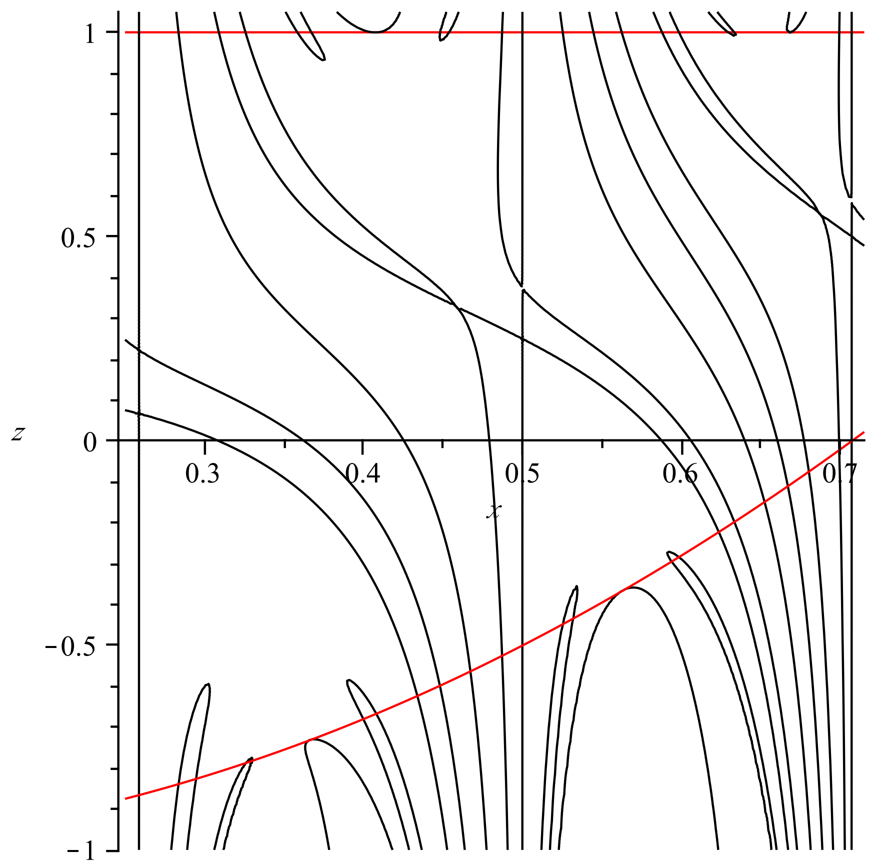

FIGURE 1. Period 2 curves in two corridors for α_{12}.

$$E = E(x, y, z) = x^2 + y^2 + z^2 - 2xyz, \tag{2.2}$$

the element $E-1$ being the trace of the element $x_1x_2x_1^{-1}x_2^{-1}$ (which, as we noted above, is $\mathrm{Aut}(F_2)$-invariant, up to conjugacy). However this turns out to be unnecessary as the action of σ_1, σ_2 on $\mathbb{Q}[x,y,z]$ is actually a representation of B_3 in $\mathrm{Aut}(\mathbb{Q}[x,y,z])$. This result is related to the fact that for any $n>1$ the braid group B_n [Bi] acts on a polynomial algebra with kernel the centre of B_n [Ma].

In general any automorphism $\phi : F_2 \to F_2$ will give rise to an automorphism of the trace ring and so determine an invertible map $\mathbb{R}^3 \to \mathbb{R}^3$. Such maps are called *trace maps* and have been studied by various authors [RB1, RB2, ABG, Ig, BR, LW, PWW]. For example in [RB1] the map $(x,y,z) \mapsto (y,z,2yz-x)$ is studied, and information is given about curves of fixed points and period doubling.

The action (2.1) of B_3 on $\mathbb{Q}[x,y,z]$ gives rise to an action of B_3 on $\mathbb{R}^3$ if we think of x,y,z as being the usual coordinate functions for $\mathbb{R}^3$. We will write this action of $\alpha \in B_3$ on $(a,b,c) \in \mathbb{R}^3$ on the right: $(a,b,c)\alpha$; this action is also the corresponding action by Nielsen transformations [MKS].

One checks that the action of B_3 fixes the function $E = E(x,y,z)$ of (2.2) and so each level set

$$E_t = \{(a,b,c) \in \mathbb{R}^3 | E(a,b,c) = t\}$$

is invariant under the action. The level set E_1 is distinguished and has been drawn by many authors [Go, RB1, RB2]. The set

$$V = \{(1,1,1), (-1,-1,1), (-1,1,-1), (1,-1,-1)\} \subset E_1$$

consists of the four singular points of E. Further, the six line segments joining these points are contained in E_1 and there is a unique component of $E_1 \setminus V$ whose closure is compact. In fact this closure is a topological 2-sphere that separates $\mathbb{R}^3$ into two components, the closure of one of these components is a 3-ball $\mathcal{T}$ that we call a "curvilinear tetrahedron". One can check that $\mathcal{T} \subset [-1,1]^3$ and that $\mathcal{T} \cap \partial[-1,1]^3$ is the above mentioned set of six line segments.

In [HM] we studied the fixed points of the diffeomorphisms

$$\alpha_n = \sigma_1^n \sigma_2^n : \mathbb{R}^3 \to \mathbb{R}^3, n > 0, 4|n.$$

We were especially interested in those fixed points which lie on $\partial\mathcal{T}$. For n even these fixed points include the points V. If we ignore the points of V for the moment, then, as pointed out in [RB1, p. 839], a consequence of the implicit function theorem is that the fixed points of α_n on $\partial\mathcal{T}$ will belong to curves of fixed points. In [HM] we described the fixed points on $\partial\mathcal{T}$; we then found the curves of fixed points which contain them and discovered which pairs of fixed points on $\partial\mathcal{T}$ are joined by these curves. We said that such a pair of fixed points is α_n-*dual*.

Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ denote the 2-torus. Then the open two-manifold $\partial\mathcal{T} \setminus V$ is covered by the restriction of the map

$$\Pi : T^2 \to \partial\mathcal{T}, \quad (\theta_1,\theta_2) \mapsto (\cos(2\pi\theta_1), \cos(2\pi\theta_2), \cos(2\pi(\theta_1+\theta_2))).$$

Note that $\Pi(\theta_1,\theta_2) = \Pi(-(\theta_1,\theta_2))$. The map Π is a branched double cover, branched over the four points V.

The action of B_3 on $\partial\mathcal{T}$ actually comes from an action of B_3 on T^2, the action being determined by the homomorphism

$$\Phi : B_3 \to SL(2,\mathbb{Z}), \qquad \sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \quad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

For $\alpha \in B_3, \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \in T^2$ the maps Π, Φ are related as follows [HM]:

$$(\Pi\theta)\alpha = \Pi(\Phi(\alpha)(\theta)). \tag{2.3}$$

The fixed points of α on $\partial\mathcal{T} \setminus V$ are of two types. First note that if $\Pi(\theta) \in \partial\mathcal{T} \setminus V \subset \mathbb{R}^3$ is fixed by α, then by (2.3) we must have $\Phi(\alpha)(\theta) = \pm\theta$. A fixed point $\Pi(\theta)$ is called α-*preserving*, or just *preserving* if α is understood, if we have $\Phi(\alpha)(\theta) = \theta$; otherwise it is called *reversing*.

§3 Curves of fixed points

In this section we recall the results of [HM] that describe the curves of fixed points for α_n.

The fixed points on $\partial\mathcal{T}$ and the fixed curves that contain them were shown to be in three families:
(F1) straight line curves;
(F2) curves in the planes $x = \pm y$;
(F3) curves not meeting the planes $x = \pm y$;

We now say a little about each of these cases:

(F1) The straight line cases. For $N \in \mathbb{N}$ let $K_N \subset SL(2,\mathbb{Z})$ denote the congruence N subgroup of $SL(2,\mathbb{Z})$, namely the kernel of the homomorphism $SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}/N\mathbb{Z})$. Note that $\Phi(\alpha_n) = \Phi(\sigma_1^n\sigma_2^n) \in K_n$. For $k, m \in \mathbb{Z}$ and any $\beta \in B_3$ such that $\Phi(\beta) \in K_n$ it follows that any point $\Pi(k/n, m/n)$ is a preserving fixed point of β. In particular, this is the case for α_n.

Now we showed [HM, Lemma 2.3] that for most integer values of k, m, n the vertical line

$$(\cos(2\pi k/n), \cos(2\pi m/n), z)$$

is a line of fixed points for α_n which contains $\Pi(k/n, m/n)$ and is not tangential to $\partial\mathcal{T}$ there. Thus this line meets $\partial\mathcal{T}$ at another point, which happens to be $\Pi(k/n, -m/n)$. Thus $\Pi(k/n, m/n)$ and $\Pi(k/n, -m/n)$ are α_n-dual.

Let $X, Y, Z \subset \mathbb{R}^3$ denote the x-axis, the y-axis and the z-axis. Now it is easily checked that any point $p \in X \cup Y \cup Z$ is fixed by each $\sigma_i^4, i = 1, 2$. Thus if n is a multiple of 4, then each of X, Y, Z is a line of fixed points for α_n which intersects $\partial\mathcal{T}$ in α_n-dual points $(\pm1, 0, 0), (0, \pm1, 0), (0, 0, \pm1)$.

(F2) The $x = \pm y$ cases. We now consider the fixed points $p = (a, b, c) \in \partial\mathcal{T}$ where $a = \pm b$. The two cases are similar and so we only describe the $a = b$ case. (In fact the map $(x, y, z) \mapsto (-x, y, -z)$ is centralised by α_n when n is even; see Lemma 4.4 below).

First note that some of the straight line curves of type (F1) are in these planes. The intersection of $\mathcal{T}$ and the plane $x = y$ is a topological disc, denoted by $\mathcal{P}$, in the $x = y$ plane bounded by the line $z = 1$ and the parabola $z = 2x^2 - 1$. In this case we showed that any such fixed point (if it is not on a vertical line of fixed points as in case (F1)) is on a curve with equation

$$\gamma^+(x) = (x, x, x(1 + U_{n-2}(x))/U_{n-1}(x)).$$

Here $U_k(x)$ is the Chebyshev polynomial of the second kind.

We draw these curves as they lie in $\mathcal{P}$ in Figure 2 for the case $n = 20$; we have shown the components of the curve $\gamma^+(x)$ as a solid curve and we have also indicated some dashed vertical lines of fixed points of type (F1) described above. We also indicate some solid vertical lines that are symmetric (see §5).

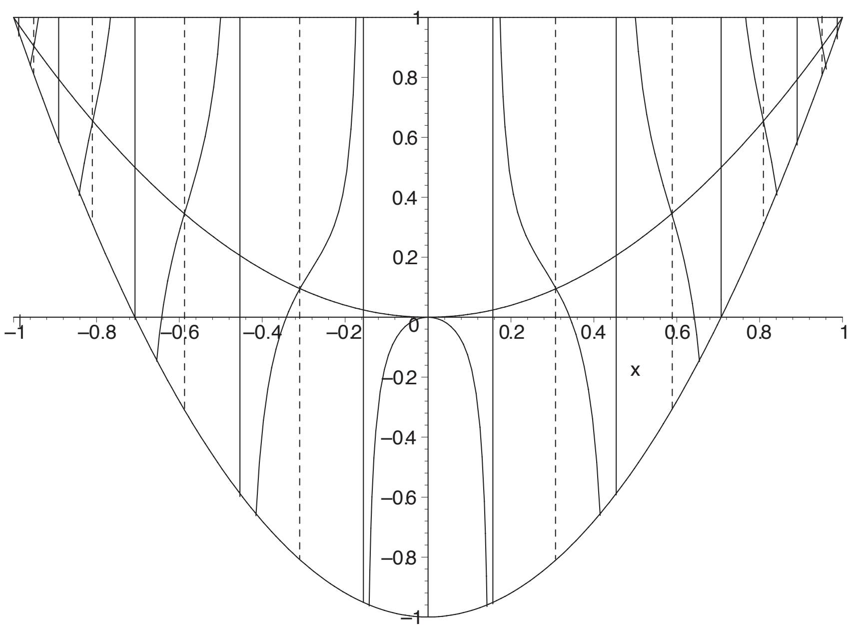

FIGURE 2. Curves of fixed points in $\mathcal{P}$ for α_{20};
dashed vertical lines are also fixed. Solid vertical lines are symmetric lines for α_{20}.

(F3) Curves not meeting the planes $x = \pm y$. We showed [HM, §§5,6] that all curves of fixed points which are not completely contained in the planes $x = \pm y$ are determined by a single polynomial $K_n(x, y)$. Further any such curve can only intersect the planes $x = \pm y$ at fixed points of type (F1) or type (F2) and these are bifurcation points. A type (F3) curve is such a curve that does not intersect the planes $x = \pm y$ at all. These will not be relevant to us in this paper.

§4 CHEBYSHEV POLYNOMIALS AND THE ACTION OF α_n

Define the Chebyshev polynomials $U_n(x)$ of the second kind [Ri] by

(4.1) $$U_{-1}(x) = 0;\ U_0(x) = 1;\ U_1(x) = 2x;\ U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x).$$

We will often need the following properties of these Chebyshev polynomials:

Lemma 4.1. [HM, Proposition 2.6] *For $m \in \mathbb{N}$ we have*
(i) $U_m(-x) = (-1)^m U_m(x)$.
(ii) $U_m(1) = m + 1$.
(iii) $U'_{2m}(0) = 0$;
(iv) $U'_{2m+1}(0) = (-1)^m 2(m + 1)$.
(v) $U_m(-1) = (-1)^m(m + 1)$.
(vi) $U^2_{m-1}(x) - U_m(x)U_{m-2}(x) = 1$.
(vii) $U^2_m(x) - 2xU_{m-1}(x)U_m(x) + U^2_{m-1}(x) = 1$.
(viii) $U_{2m}(x) = U^2_m(x) - U^2_{m-1}(x)$.

(ix) $U_{2m-1}(x) = 2U_m(x)U_{m-1}(x) - 2xU_{m-1}^2(x)$.
(x) For all even $n > 1$ *we have* $\gcd(1 + U_{n-2}(x), U_{n-1}(x)) = U_{n/2-1}(x)$.
(xi) $U_m'(1) = 2\binom{m+2}{3}$.

The Chebyshev polynomials of the first kind are defined as follows:

$$T_{-1}(x) = 0;\ T_0(x) = 1;\ T_1(x) = x;\ T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

Lemma 4.2. [Ri] $U_n'(x) = \frac{(n+1)T_{n+1}(x) - xU_n(x)}{x^2-1}$.

Lemma 4.3. [HM, Lemma 2.1] *If* $k \in \mathbb{Z}$, *then*

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 2y \end{pmatrix}^k \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \sigma_1^k;$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2x & -1 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \sigma_2^k.$$

Denote the two matrices above by $N_1 = N_1(y), N_2 = N_2(x)$. *Then we have:*

$$N_1^k = \begin{pmatrix} -U_{k-2}(y) & 0 & U_{k-1}(y) \\ 0 & 1 & 0 \\ -U_{k-1}(y) & 0 & U_k(y) \end{pmatrix};$$

$$N_2^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & U_k(x) & -U_{k-1}(x) \\ 0 & U_{k-1}(x) & -U_{k-2}(x) \end{pmatrix}.$$

Lemma 4.4. [HM, Proposition 2.5] *(i) The involutive automorphism*

$$S : \mathbb{Q}[x, y, z] \to \mathbb{Q}[x, y, z], \quad (x, y, z) \mapsto (-x, y, -z)$$

centralises any $\alpha \in \langle \sigma_1, \sigma_2^2 \rangle$, *that is* $\alpha S = S\alpha$. *In particular, if* n *is even, then* S *centralises* α_n.

(ii) The involutive automorphism

$$R : \mathbb{Q}[x, y, z] \to \mathbb{Q}[x, y, z], \quad (x, y, z) \mapsto (y, x, z)$$

conjugates σ_1 *to* σ_2^{-1}. *The map* R *reverse centralises* α_n *so that* $\alpha_n R = R\alpha_n^{-1}$.

(iii) The involutive automorphism

$$T = (SR)^2 : \mathbb{Q}[x, y, z] \to \mathbb{Q}[x, y, z], (x, y, z) \mapsto (-x, -y, z)$$

commutes with σ_1^2 *and with* σ_2^2. *In particular, if* n *is even, then* T *centralises* α_n.

We note that the involution S maps the $x = y$ plane bijectively onto the $x = -y$ plane. Lemma 4.4 (i) thus determines a correspondence between fixed and period 2 points in the respective planes. Thus it suffices for us to study $\mathcal{P}$. Also Lemma 4.4 (iii) shows that the fixed and period 2 points in $\mathcal{P}$ are symmetric relative to Z, as seen in Figure 2.

Lemma 4.5. [HM, Corollary 2.2] *For all* $n \in \mathbb{N}$ *and* $(x, y, z)^T \in \mathbb{R}^3$ *we have:*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \alpha_n = \begin{pmatrix} -xU_{n-2}(y) + zU_{n-1}(y) \\ U_n(x^*)y - U_{n-1}(x^*)[-xU_{n-1}(y) + zU_n(y)] \\ U_{n-1}(x^*)y - U_{n-2}(x^*)[-xU_{n-1}(y) + zU_n(y)] \end{pmatrix}.$$

Here $x^* = -xU_{n-2}(y) + zU_{n-1}(y)$.

In particular, if $(x, y, z)^T \in \mathbb{R}^3$ is a fixed point of α_n and $U_{n-1}(y) \neq 0$, then we must have

$$z = x(1 + U_{n-2}(y))/U_{n-1}(y).$$

Lemma 4.6. *The point (x, x, x^2), where $x = \cos\frac{2\pi j}{n}$ is on the curve $\gamma^+(t) = (t, t, \frac{t(1+U_{n-2}(t))}{U_{n-1}(t)})$ of fixed points for α_n.*

Proof In [HM, Lemma 3.2 (i)] we showed that $\gamma^+(t)$ is a curve of fixed points for α_n.

Let $x = \cos\frac{2\pi j}{n}$. We need to show that $\frac{x(1+U_{n-2}(x))}{U_{n-1}(x)} = x^2$. Now $U_{n-1}(x) = \frac{\sin n\frac{2\pi j}{n}}{\sin\frac{2\pi j}{n}} = 0$, and $U_{n-2}(x) = \frac{\sin(n-1)\frac{2\pi j}{n}}{\sin\frac{2\pi j}{n}} = -1$. Further, $T_n(x) = \cos n\frac{2\pi j}{n} = 1$, and $T_{n-1}(x) = \cos(n-1)\frac{2\pi j}{n} = x$. Using Lemma 4.2 we obtain

$$\begin{aligned}\frac{x(1+U_{n-2}(x))}{U_{n-1}(x)} &= \frac{x\frac{d}{dx}[1+U_{n-2}(x))]}{\frac{d}{dx}U_{n-1}(x)}\\ &= \frac{x[(n-1)T_{n-1}(x) - xU_{n-2}(x)]}{nT_n(x) - xU_{n-1}(x)}\\ &= \frac{x[(n-1)T_{n-1}(x) + x]}{nT_n(x)}\\ &= x^2,\end{aligned}$$

as required. □

§5 General results about curves of period 2 points in $\mathcal{P}$

Theorem 5.1. *Let $p \in \mathcal{P}$. Then $p \in \mathcal{P} \cap (\mathcal{P})\alpha_n$ if and only if $(p)\alpha_n^2 = p$.*

Proof Let $p = (x, x, z) \in \mathcal{P}$ and assume that $p \in \mathcal{P} \cap (\mathcal{P})\alpha_n$ From Lemma 4.3 we have:

$$\begin{aligned}(x, x, z)\alpha_n &= (v_1, v_2, v_3)\\ &= (-xU_{n-2}(x) + zU_{n-1}(x),\\ &\qquad U_n(x^*)x - U_{n-1}(x^*)(-U_{n-1}(x)x + zU_n(x)),\\ &\qquad xU_{n-1}(x^*) - U_{n-2}(x^*)(-U_{n-1}(x)x + zU_n(x))).\end{aligned} \tag{5.1}$$

Here $x^* = v_1 = -xU_{n-2}(x) + zU_{n-1}(x)$. From the same result we also have:

$$\begin{aligned}(x, x, z)\alpha_n^{-1} &= (u_1, u_2, u_3)\\ &= (xU_n(x^*) - U_{n-1}(x^*)\left(-xU_{n-1}(x) + zU_n(x)\right),\\ &\qquad -xU_{n-2}(x) + zU_{n-1}(x),\\ &\qquad xU_{n-1}(x^*) - U_{n-2}(x^*)\left(-xU_{n-1}(x) + zU_n(x)\right)).\end{aligned} \tag{5.2}$$

From $p \in \mathcal{P} \cap (\mathcal{P})\alpha_n$ we see that $(p)\alpha_n^{-1} \in \mathcal{P}$. This shows that $u_1 = u_2$. But we have $x^* = v_1 = u_2$, and from (5.1) and (5.2) we see that $v_2 = u_1$, so that $u_1 = u_2 = v_1 = v_2$. It is also clear from (5.1) and (5.2) that $u_3 = v_3$, so that $(x, x, z)\alpha_n = (x, x, z)\alpha_n^{-1}$, giving $(p)\alpha_n^2 = p$.

Now if $(p)\alpha_n^2 = p$ for $p = (x, x, z) \in \mathcal{P}$, then we have $(p)\alpha_n = (p)\alpha_n^{-1}$. Equating the entries in (5.1) and (5.2) we see that $v_1 = u_1, v_2 = u_2, v_3 = u_3$, but we also have $v_1 = u_2$, so that $u_1 = u_2 = v_1 = v_2$, showing that $(p)\alpha_n \in \mathcal{P}$. □

Remarks 1. In torus coordinates α_n maps $\partial\mathcal{P}$ by $M_n = \begin{pmatrix} 1 & n \\ -n & 1-n^2 \end{pmatrix}$ which has trace $2 - n^2 < -2$, so each point of $(\partial\mathcal{P} \setminus V) \cap \mathrm{Fix}(\alpha_n^2)$ is a hyperbolic periodic point of $\alpha_n|_{\partial\mathcal{P}}$ and, by the Implicit Function Theorem (as noted in [RB1, p. 839]), belongs to a smooth curve in $\mathrm{Fix}(\alpha_n^2)$ that is transverse there to $\partial\mathcal{T}$.

2. Consider $G : \mathcal{P} \to \mathbb{R}, G(x, x, z) := v_1 - v_2$ where $(x, x, z)\alpha_n = (v_1, v_2, v_3)$. Then $\{G = 0\} = \mathcal{P} \cap (\mathcal{P})\alpha_n$. From Algebraic Geometry, see §2.3 of [W], $G = 0$ on a finite union of curves, each parametrised by a Puiseux series, so $\mathcal{P} \cap (\mathcal{P})\alpha_n$ is certainly a union of smooth curves, possibly with cusps (and there may also be some isolated points). Now $G = 0$ at each point $p \in \partial\mathcal{P} \cap (\partial\mathcal{P})\alpha_n = \partial\mathcal{P} \cap \mathrm{Fix}(\alpha_n^2)$. These points are parametrised by points of the straight line joining $(0, 0)$ to $(1, 1)$ or $(1, -1)$ that are mapped by M_n to one of these lines (mod $\mathbb{Z}^2$), and we note that the images of these lines do cross the lines there. Thus $G|\partial\mathcal{P}$ changes sign at each point of $\partial\mathcal{P} \cap (\partial\mathcal{P})\alpha_n$. The component of $\{G = 0\}$ to which p belongs must meet $\partial\mathcal{P}$ in at least one more point since otherwise it would not be able to separate the points where $G > 0$ near p from those where $G < 0$. We say that p is *dual* to $q \in \partial\mathcal{P} \cap \mathrm{Fix}(\alpha_n^2)$ if p and q are endpoints of the intersection with $\mathcal{P}$ of a smooth curve in $\mathrm{Fix}(\alpha_n^2)$. We shall discover which of the various points of $\partial\mathcal{P}$ in this component p is dual to in Theorem 7.5.

§6 Symmetric curves of period 2 points in $\mathcal{P}$

We call a point $(x, x, z) \in \mathbb{R}^3$ a *symmetric point* (for α_n) if $(x, x, z)\alpha_n = (-x, -x, z)$. Such a point is a point of period 2 by Lemma 4.4 (iii).

Symmetric points (and curves of such points) are determined in the following result, which also determines duality for symmetric points.

Proposition 6.1. *(i) The curve* $\gamma^-(x) = \left(x, x, \frac{xU_{2n-2}(x)}{U_{2n-1}(x)}\right)$ *is fixed by* α_{4n}^2.

(ii) The vertical line $(x, x, z), x = \cos\frac{k\pi}{4n}$*, is also fixed by* α_{4n}^2 *when* k *is odd.*

(iii) The curves in (i) and (ii) are exactly the symmetric curves.

(iv) The symmetric curves $\gamma^-(x)$ *meet* $\mathcal{P}$ *in components that intersect* $\partial\mathcal{T}$ *whenever* $x = \cos 2\pi\theta$*, where*

$$\theta = \frac{k}{2(4n-2)} \text{ and } \theta = \frac{k}{2(4n+2)} \text{ for } k \text{ odd.}$$

The points with denominator $2(4n-2)$ *are on* $\partial^+\mathcal{P}$*, while the points with denominator* $2(4n+2)$ *are on* $\partial^-\mathcal{P}$*. See Figure 3.*

(v) For each odd k*, the curve* $\gamma^-(x)$ *joining the points of* $\partial\mathcal{P}$ *where* $\theta = \frac{k}{2(4n-2)}$ *and* $\theta = \frac{k}{2(4n+2)}$*, meets the given vertical line (see (ii) above) at* $(x, x, x^2), x = \cos\frac{2k\pi}{8n}$*. This is the point on the line where* $E = x^2 + x^2 + z^2 - 2x^2z$ *takes its minimum value.*

Proof Let $U_k = U_k(x)$.

(i) Now $\alpha_{4n} = \sigma_1^{4n}\sigma_2^{4n}$ and so by Lemma 4.3 we have

$$(x, x, \frac{xU_{2n-2}}{U_{2n-1}})^T \sigma_1^{4n} = \begin{pmatrix} -U_{4n-2} & 0 & U_{4n-1} \\ 0 & 1 & 0 \\ -U_{4n-1} & 0 & U_{4n} \end{pmatrix} \begin{pmatrix} x \\ x \\ \frac{xU_{2n-2}}{U_{2n-1}} \end{pmatrix}$$

$$= \begin{pmatrix} -xU_{4n-2} + \frac{xU_{2n-2}U_{4n-1}}{U_{2n-1}} \\ x \\ -xU_{4n-1} + \frac{xU_{2n-2}U_{4n}}{U_{2n-1}} \end{pmatrix}. \tag{6.1}$$

Before proceeding to the action of σ_2^{4n} we simplify (6.1). For the first entry of (6.1) we have, using Lemma 4.1 :

$$\begin{aligned} -xU_{4n-2} + \frac{xU_{2n-2}U_{4n-1}}{U_{2n-1}} &= \frac{-xU_{4n-2}U_{2n-1} + xU_{2n-2}U_{4n-1}}{U_{2n-1}} \\ &= \frac{-xU_{2n-1}(U_{2n-1}^2 - U_{2n-2}^2) + xU_{2n-2}(2U_{2n}U_{2n-1} - 2xU_{2n-1}^2)}{U_{2n-1}} \\ &= x(-U_{2n-1}^2 + U_{2n-2}^2 + 2U_{2n}U_{2n-2} - 2xU_{2n-1}U_{2n-2}) \\ &= x(-U_{2n-1}^2 + U_{2n-2}^2 + 2(2xU_{2n-1} - U_{2n-2})U_{2n-2} - 2xU_{2n-1}U_{2n-2}) \\ &= x(-U_{2n-1}^2 - U_{2n-2}^2 + 2xU_{2n-1}U_{2n-2}) \\ &= -x, \end{aligned}$$

where the last equality is given by Lemma 4.1 (vii).

For the third entry of (6.1) we have:

$$\begin{aligned} -xU_{4n-1} + \frac{xU_{2n-2}U_{4n}}{U_{2n-1}} &= \frac{-xU_{4n-1}U_{2n-1} + xU_{2n-2}U_{4n}}{U_{2n-1}} \\ &= \frac{-x(2U_{2n}U_{2n-1} - 2xU_{2n-1}^2)U_{2n-1} + xU_{2n-2}(U_{2n}^2 - U_{2n-1}^2)}{U_{2n-1}}. \end{aligned}$$

Now substitute $U_{2n} = 2xU_{2n-1} - U_{2n-2}$ into this and the resulting equation factors:

$$\frac{x(U_{2n-2} - 2xU_{2n-1})(U_{2n-1}^2 - 2xU_{2n-1}U_{2n-2} + U_{2n-2}^2)}{U_{2n-1}}.$$

The last factor of the numerator is 1 by Lemma 4.1 (vii). Thus this expression is

$$\frac{x(U_{2n-2} - 2xU_{2n-1})}{U_{2n-1}} = \frac{xU_{2n-2}}{U_{2n-1}} - 2x^2.$$

Thus (6.1) is equal to

$$\begin{pmatrix} -x \\ x \\ \frac{xU_{2n-2}}{U_{2n-1}} - 2x^2 \end{pmatrix}.$$

We now act on this by σ_2^{4n}, again using Lemma 4.3, where we recall that $U_k(-x) = -U_k(x)$ if k is odd and $U_k(-x) = U_k(x)$ if not:

$$\begin{pmatrix} -x \\ x \\ \frac{xU_{2n-2}}{U_{2n-1}} - 2x^2 \end{pmatrix} \sigma_2^{4n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & U_{4n} & U_{4n-1} \\ 0 & -U_{4n-1} & -U_{4n-2} \end{pmatrix} \begin{pmatrix} -x \\ x \\ \frac{xU_{2n-2}}{U_{2n-1}} - 2x^2 \end{pmatrix}$$
$$= \begin{pmatrix} -x \\ xU_{4n} + U_{4n-1}\left(\frac{xU_{2n-2}}{U_{2n-1}} - 2x^2\right) \\ -xU_{4n-1} - U_{4n-2}\left(\frac{xU_{2n-2}}{U_{2n-1}} - 2x^2\right) \end{pmatrix}.$$

For the second entry we substitute for U_{4n}, U_{4n-1} (using Lemma 4.1) and $U_{2n} = 2xU_{2n-1} - U_{2n-2}$ and find that the resulting expression becomes:

$$-x(U_{2n-1}^2 - 2xU_{2n-1}U_{2n-2} + U_{2n-2}^2) = -x.$$

Doing the same thing for the third entry we obtain $\frac{xU_{2n-2}}{U_{2n-1}}$, so that we now have

$$\left(x, x, \frac{xU_{2n-2}(x)}{U_{2n-1}(x)}\right)\sigma_1^{4n}\sigma_2^{4n} = \left(-x, -x, \frac{xU_{2n-2}(x)}{U_{2n-1}(x)}\right).$$

Since $\frac{xU_{2n-2}(x)}{U_{2n-1}(x)}$ is an even function, it follows that $(\sigma_1^{4n}\sigma_2^{4n})^2$ fixes $\left(x, x, \frac{xU_{2n-2}(x)}{U_{2n-1}(x)}\right)$ and we have proved (i).

(ii) Now, if k is odd and $x = \cos 2\pi\theta, \theta = \frac{k}{8n}$, then $U_{4n-1}(x) = \frac{\sin 8n\pi\theta}{\sin 2\pi\theta} = 0$. We also have $U_{4n-2}(x) = \frac{\sin(4n-1)2\pi\theta}{\sin 2\pi\theta} = 1$ and $U_{4n}(x) = \frac{\sin(4n+1)\theta 2\pi}{\sin \theta 2\pi} = -1$, so that, by Lemma 4.3 we have

$$N_1^{4n} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad N_2^{4n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

so that

$$N_2^{4n}N_1^{4n} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus $(x, x, z)\alpha_{4n} = (-x, -x, z)$ and so

$$(x, x, z)\alpha_{4n}^2 = (x, x, z).$$

These vertical lines are drawn solid in Figure 2 for $n = 20$.

(iii) From Lemma 4.5 we see that a symmetric point $(x, x, z)^T$ for α_{4n} must satisfy $-x = x^* = -xU_{4n-2}(x) + zU_{4n-1}(x)$, so that using Lemma 4.1 we have:

$$\begin{aligned} z &= \frac{-x(1 - U_{4n-2})}{U_{4n-1}} \\ &= \frac{-x(1 - U_{2n-1}^2 + U_{2n-2}^2)}{2U_{2n}U_{2n-1} - 2xU_{2n-1}^2} \\ &= \frac{-x(U_{2n-2}^2 - 2xU_{2n-1}U_{2n-2} + U_{2n-2}^2)}{2U_{2n-1}(U_{2n} - xU_{2n-1})} \\ &= \frac{xU_{2n-2}}{U_{2n-1}}. \end{aligned}$$

This shows that a symmetric point with $U_{2n-1}(x) \neq 0$ must be $\gamma^-(x)$. If $U_{2n-1}(x) = 0$, then $x = \cos 2\pi\theta$ where $\sin(2n2\pi\theta) = 0$; this gives $\theta = \frac{k}{4n}$, however for k even these vertical lines are fixed, while if k is odd they are symmetric by (ii).

(iv) Let $z = \frac{xU_{2n-2}(x)}{U_{2n-1}(x)}$. Then the expression $E(x,x,z)-1 = x^2+x^2+z^2-2x^2z-1$ factors as

$$\frac{(U_{2n-1} - xU_{2n-2})(2x^2U_{2n-1} - U_{2n-1} - xU_{2n-2})}{U_{2n-1}^2}.$$

Substituting $U_k = \frac{\sin(k+1)\theta 2\pi}{\sin\theta 2\pi}$ (where $x = \cos\theta 2\pi$) the first factor reduces to

$$\frac{\sin(2n\theta 2\pi) - \cos\theta 2\pi \sin(2n-1)\theta 2\pi}{\sin\theta 2\pi} = \cos(2n-1)\theta 2\pi. \tag{6.2}$$

The solutions in this case are $\theta = \frac{k}{4(2n-1)}$, for odd k.

Now when $\theta = \frac{k}{4(2n-1)}$ with odd k and $x = \cos(\theta 2\pi)$, then we have

$$z(x) = \frac{xU_{2n-2}}{U_{2n-1}} = \frac{\cos(\frac{k\pi}{2(2n-1)})\sin(\frac{(2n-1)k\pi}{2(2n-1)})}{\sin(\frac{2nk\pi}{2(2n-1)})}.$$

Now if we put $\theta = \frac{k}{4(2n-1)}$ into (6.2), then the right hand side is 0 and the left hand side shows that $\sin(2n\theta 2\pi) - \cos\theta 2\pi\sin(2n-1)\theta 2\pi = 0$ and so the above expression for $z(x)$ is 1. Thus these points are on $\partial^+\mathcal{P}$.

Similarly, the second factor is

$$\begin{aligned}
&\frac{2\cos^2\theta 2\pi\sin(2n\theta 2\pi) - \sin(2n\theta 2\pi) - \cos(\theta 2\pi)\sin(2n-1)\theta 2\pi)}{\sin\theta 2\pi}\\
&= \frac{2\cos^2\theta 2\pi\sin(2n\theta 2\pi) - \sin(2n\theta 2\pi) - \cos(\theta 2\pi)(\sin 2n\theta 2\pi\cos\theta 2\pi - \sin\theta 2\pi\cos(2n)\theta 2\pi)}{\sin\theta 2\pi}\\
&= \frac{\sin 2n\theta 2\pi - \cos\theta 2\pi\sin\theta 2\pi\cos(2n)\theta 2\pi - \cos^2\theta 2\pi\sin 2n\theta 2\pi}{\sin\theta 2\pi}\\
&= \frac{\sin^2\theta 2\pi\sin 2n\theta 2\pi - \cos\theta 2\pi\sin\theta 2\pi\cos(2n)\theta 2\pi}{\sin\theta 2\pi}\\
&= -\cos(2n+1)\theta 2\pi.
\end{aligned}$$

The solutions in this case are $\theta = \frac{k}{4(2n+1)}$, for odd k.

One now shows that $z(x) = \cos(2\theta 2\pi)$ if $\theta = \frac{k}{4(2n+1)}$, for odd k and $x = \cos\theta 2\pi$, so that these points are on $\partial^-\mathcal{P}$.

(v) If k is odd and $x = \cos\frac{k\pi}{4n}$, then

$$\begin{aligned}
z(x) = \frac{xU_{2n-2}(x)}{U_{2n-1}(x)} &= \frac{\cos\frac{k\pi}{4n}\sin\frac{(2n-1)k\pi}{4n}}{\sin\frac{k\pi}{2}}\\
&= \frac{\cos\frac{k\pi}{4n}\left(\sin\frac{k\pi}{2}\cos\frac{k\pi}{4n} - \cos\frac{k\pi}{2}\sin\frac{k\pi}{4n}\right)}{\sin\frac{k\pi}{2}}\\
&= \cos^2\frac{k\pi}{4n} = x^2.
\end{aligned}$$

That this is the point on the vertical line where $x^2+x^2+z^2-2x^2z$ takes its minimum value was shown in [HM, Lemma 2.8]. $\square$

From Proposition 6.1 we see that the curve

$$\gamma^-(x) = (x,x,z), \quad z = z(x) = \frac{xU_{2n-2}(x)}{U_{2n-1}(x)},$$

is a curve of period two points for α_{4n} that has the property that $(x,x,z)\alpha_{4n} = (-x,-x,z)$. Thus we have $\gamma^-(x)\alpha_{4n} \neq \gamma^-(x)$, whenever $x \neq 0$.

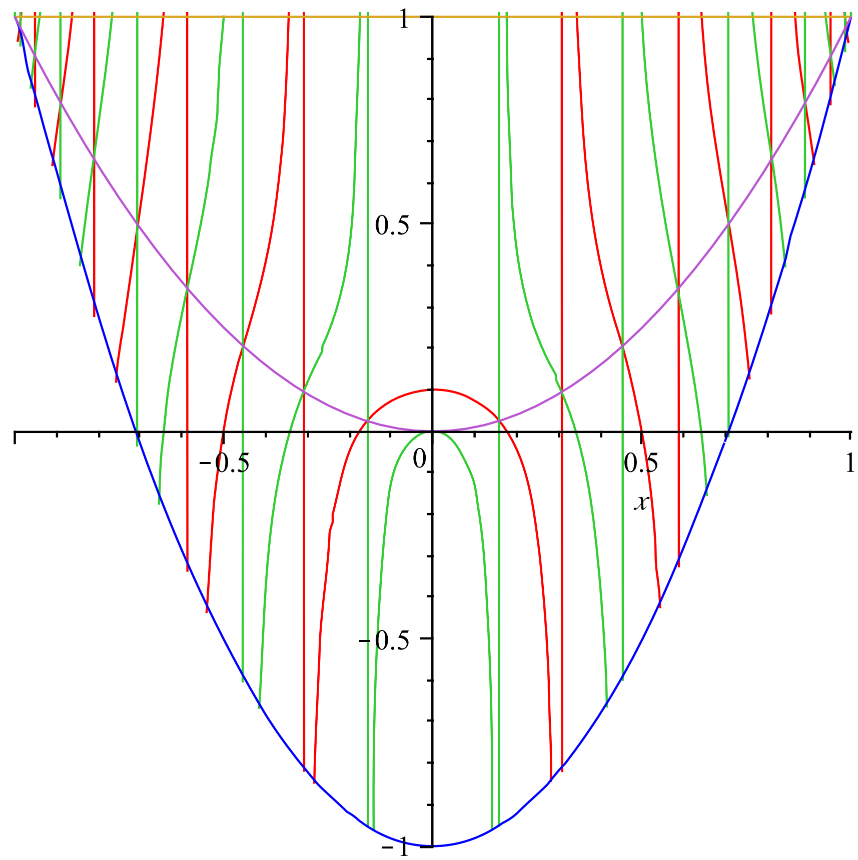

FIGURE 3. Symmetric curves and fixed curves for α_{20}

We illustrate this in Figure 3 for $n = 20$. Compare Figure 3 with Figure 2.

Let $\gamma^+(x) = (x, x, \frac{x(1+U_{4n-2})}{U_{4n-1}})$ be the curve of fixed points of α_{4n}, as given in Lemma 4.6. From the first statement of [HM, Proposition 3.5] we see that the points of $\gamma^+(x) \cap \partial\mathcal{P}$ are

$$\cos\frac{k\pi}{4n+2}, \text{ for } k = 0, 2, 4, \dots, 4n+2;$$
$$\cos\frac{k\pi}{4n-2}, \text{ for } k = 0, 2, 4, \dots, 4n-2.$$

It follows that the points of $\gamma^-(x) \cap \partial\mathcal{P}$ alternate with the points of $\gamma^+(x) \cap \partial\mathcal{P}$ on each of $\partial^+\mathcal{P}$ and $\partial^-\mathcal{P}$. Since for $x \neq 0$ the points of $\gamma^-(x)$ are not fixed and the points of $\gamma^+(x)$ are fixed by α_{4n} we see that the curves $\gamma^+(x)$ and $\gamma^-(x)$ are disjoint. Now from the above we see that between consecutive fixed curves of $\gamma^+(x) \cap \mathcal{P}$ there are exactly two points of $\gamma^-(x) \cap \partial\mathcal{P}$. These thus must be dual points. This gives most of

Theorem 6.2. *For $1 \le k \le 2n-3$ odd, the points with torus coordinates $(\frac{k}{4n-2}, -\frac{k}{4n-2})$ and $(\frac{k}{4n+2}, \frac{k}{4n+2})$ are dual symmetric points for α_{4n}.*

The points with coordinates $(\frac{2n-1}{4n+2}, \frac{2n-1}{4n+2})$ and $(\frac{2n+3}{4n+2}, \frac{2n+3}{4n+2})$ are dual symmetric points for α_{4n}. They are joined by a symmetric curve of points that passes through the point $(0, 0, \frac{1}{2n})$.

The points with coordinates $(1 - \frac{k}{4n-2}, -1 + \frac{k}{4n-2})$ and $(1 - \frac{k}{4n+2}, 1 - \frac{k}{4n+2})$ are dual when $1 \le k \le 2n-3$ and k is odd.

Proof We need only be concerned about the second paragraph. Clearly these points are dual. Now the curve $\gamma^-(x)$ goes through the point $(0, 0, z_0)$, where

$$z_0 = \frac{\frac{d}{dx} x U_{2n-2}(x)}{\frac{d}{dx} U_{2n-1}(x)} = \frac{(U_{2n-2}(x) + x\frac{(2n-1)T_{2n-1}(x) - xU_{2n-2}(x)}{x^2-1})|_{x=0}}{\frac{2nT_{2n}(x) - xU_{2n-1}(x)}{x^2-1}|_{x=0}}.$$

Letting x go to zero gives $\frac{U_{2n-2}(0)}{-2nT_{2n}(0)} = \frac{1}{2n}$. Here we use that fact that $U_n(0) = T_n(0) = (-1)^{\frac{n}{2}}$ if n is even, and $U_n(0) = T_n(0) = 0$ if n is odd. $\square$

We note that this means that the trace of the Jacobian of α_{4n} at the fixed point $(0, 0, z_0)^T$ must be $1 - 1 - 1 = -1$.

§7 Duality, Stalagmites, Stalactites and Columns

Our aim in this section is to determine the dual pairing of all the points fixed by α_n^2 in $\partial\mathcal{P}$. We emphasize that we will be assuming $4|n$. First we collect the results already proved.

Proposition 7.1. *(i) For k even in $\{1, \dots, n-1\}$ the vertical line $(\cos(2\pi k/(2n)), \cos(2\pi k/(2n)), z)$ consists of points fixed by α_n.*

(ii) For k odd in $\{1, \dots, n-1\}$ the vertical line $(\cos(2\pi k/(2n)), \cos(2\pi k/(2n)), z)$ consists of symmetric points fixed by α_n^2. Recall that (x, x, z) is called symmetric if $(x, x, z)\alpha_n = (-x, -x, z)$ and then it has period 2 because also $(-x, -x, z)\alpha_n = (x, x, z)$.

(iii) The curve $\gamma^+(x) = (x, x, x(1 + U_{n-2}(x))/U_{n-1}(x))$ consists of fixed points and, for k even in $\{1, \dots, n/2-2\}$, it joins the dual pair corresponding to the parameters $k/(2(n+2))$ in $\partial^-\mathcal{P}$ and $k/(2(n-2))$ in $\partial^+\mathcal{P}$ and the dual pair corresponding to the parameters $1/2 - k/(2(n+2))$ in $\partial^-\mathcal{P}$ and $1/2 - k/(2(n-2))$ in $\partial^+\mathcal{P}$; it also joins the dual pair in $\partial^-\mathcal{P}$ corresponding to the parameters $(n/2)/(2(n+2)), (n/2+2)/(2(n+2))$ (which are $1/4 \pm 1/(2(n+2))$).

(iv) The curve $\gamma^-(x) = (x, x, xU_{n/2-2}(x)/U_{n/2-1}(x))$ *consists of symmetric points and, for* k *odd in* $\{1, \ldots, n/2-2\}$*, it joins the dual pair corresponding to the parameters* $k/(2(n+2))$ *in* $\partial^-\mathcal{P}$ *and* $k/(2(n-2))$ *in* $\partial^+\mathcal{P}$ *and their images which are the dual pair corresponding to the parameters* $1/2 - k/(2(n+2))$ *in* $\partial^-\mathcal{P}$ *and* $1/2 - k/(2(n-2))$ *in* $\partial^+\mathcal{P}$*; also it joins the dual pair in* $\partial^-\mathcal{P}$ *with parameters* $(n/2-1)/(2(n+2))$ *and* $(n/2+3)/(2(n+2))$ *(which are* $1/4 \pm 1/(n+2)$*).*

Proof (i) was Lemma 2.12 of [HM] and (iii) was Lemma 3.5 of [HM]; (ii) and (iv) were Proposition 6.1 and Theorem 6.2 above. □

We divide $\mathcal{P}$ into n distinct (open) vertical *corridors* separated by the vertical lines of fixed and symmetric points of Proposition 7.1 (i) and (ii). For $k \in \mathbb{Z}$ we specify the kth corridor by

$$C_k := \{(\cos(2\pi\theta), \cos(2\pi\theta), z) \in \mathcal{P} : k/(2n) < \theta < (k+1)/(2n)\}$$

and note that $C_{2n+k} = C_k = C_{-k-1}$. We let $D_k := (k/(2n), (k+1)/(2n))$ parametrise the top and bottom edges

$$C_k^+ := C_k \cap \partial^+\mathcal{P}, \quad C_k^- := C_k \cap \partial^-\mathcal{P}$$

of C_k using

$$\theta \mapsto (\cos(2\pi\theta), \cos(-2\pi\theta), 1),\ (\cos(2\pi\theta), \cos(2\pi\theta), \cos(4\pi\theta)).$$

By studying which corridor contains the image of a point of period 2 we shall determine the duality. One point in $\partial^+\mathcal{P}$ may be dual to another there; then we call the curve of period 2 points joining them a *stalactite*. If it is dual to a point of $\partial^-\mathcal{P}$, then we call the curve a *column*. If one point of $\partial^-\mathcal{P}$ is dual to another, then we call such a curve a *stalagmite*.

We will show that the image under α_n of a stalagmite or a stalactite is a column, and that all columns are either (i) curves of fixed points; (ii) symmetric curves of period 2; or (iii) have images under α_n that are either a stalagmite or a stalactite.

Now α_n acts on C_k^+ and C_k^- by multiplying the parameter in D_k by $(n-1)$ and $(n+1)$ respectively, as we see from the first coordinate (or its negative) of

$$\begin{pmatrix} 1 & n \\ -n & 1-n^2 \end{pmatrix} \begin{pmatrix} \theta \\ -\theta \end{pmatrix} = -\begin{pmatrix} (n-1)\theta \\ (1+n-n^2)\theta \end{pmatrix},$$

$$\begin{pmatrix} 1 & n \\ -n & 1-n^2 \end{pmatrix} \begin{pmatrix} \theta \\ \theta \end{pmatrix} = \begin{pmatrix} (n+1)\theta \\ (1-n-n^2)\theta \end{pmatrix}.$$

The points of $\partial^+\mathcal{P} \cap (\partial^+\mathcal{P})\alpha_n^{-1}$ and $\partial^+\mathcal{P} \cap (\partial^-\mathcal{P})\alpha_n^{-1}$ have parameters

$$\{j/(n(n-2)) : 0 \le j \le n(n-2)/2\},\ \{j/(n^2-2) : 0 \le j \le (n^2-2)/2\}.$$

In fact

$$\begin{pmatrix} 1 & n \\ -n & 1-n^2 \end{pmatrix} \begin{pmatrix} j/(n(n-2)) \\ -j/(n(n-2)) \end{pmatrix} = \begin{pmatrix} 0 \\ -j \end{pmatrix} + (1-n) \begin{pmatrix} j/(n(n-2)) \\ -j/(n(n-2)) \end{pmatrix},$$

while

$$\begin{pmatrix} 1 & n \\ -n & 1-n^2 \end{pmatrix} \begin{pmatrix} j/(n^2-2) \\ -j/(n^2-2) \end{pmatrix} = \begin{pmatrix} 0 \\ -j \end{pmatrix} + (1-n) \begin{pmatrix} j/(n^2-2) \\ j/(n^2-2) \end{pmatrix}.$$

Similarly the points of $\partial^-\mathcal{P} \cap (\partial^+\mathcal{P})\alpha_n^{-1}$ and $\partial^-\mathcal{P} \cap (\partial^-\mathcal{P})\alpha_n^{-1}$ have parameters

$$\{j/(n^2-2) : 0 \le j \le (n^2-2)/2\},\ \{j/(n(n+2)) : 0 \le j \le n(n+2)/2\}.$$

For $0 \le k < n$ the set of parameters of points in $C_k^+ \cap (\partial^+\mathcal{P})\alpha_n^{-1}$ is

$$\{\theta_{k,j}^{++} := (k(n/2-1)+j)/(n(n-2)) : 1 \le j \le n/2-2\},$$

while the set of parameters of points in $C_k^- \cap (\partial^-\mathcal{P})\alpha_n^{-1}$ is

$$\{\theta_{k,j}^{--} := (k(n/2+1)+j)/(n(n+2)) : 1 \le j \le n/2\}.$$

Thus $\theta_{k,j}^{++}$ is the parameter of the jth point of C_k^+ that is mapped by α_n from $\partial^+\mathcal{P}$ to $\partial^+\mathcal{P}$. Notice that putting $j=0$ or $n/2-1$ in $\theta_{k,j}^{++}$ (or putting $j=0$ or $n/2+1$ in $\theta_{k,j}^{--}$) would give a fixed or symmetric point as in Proposition 7.1 (i) or (ii) and these are in the boundary of the corridor C_k.

Also, for $0 \le k < n$, the set of parameters of points in $C_k^+ \cap (\partial^-\mathcal{P})\alpha_n^{-1}$ is

$$\{\theta_{k,j}^{+-} := (kn/2+j)/(n^2-2) : 0 \le j < n/2\},$$

while the set of parameters of points in $C_k^- \cap (\partial^+\mathcal{P})\alpha_n^{-1}$ is similarly

$$\{\theta_{k,j}^{-+} := (kn/2+j)/(n^2-2) : 0 \le j < n/2\},$$

except that we exclude $\theta_{0,0}^{+-} = \theta_{0,0}^{-+} = 0$ and $\theta_{n-1,n/2-1}^{+-} = \theta_{n-1,n/2-1}^{-+} = \frac{1}{2}$.

Proposition 7.2. *If $0 \le k < n$ and k is even then*
(i) α_n maps the point corresponding to $\theta_{k,j}^{++}$ to a point in $C_{2j-k} = C_{k-2j-1}$ for $1 \le j \le n/2-2$.
(ii) α_n maps the point corresponding to $\theta_{k,j}^{+-}$ to a point in $C_{2j-k} = C_{k-2j-1}$ if $0 \le j \le k$ and in C_{2j-k-1} if $k < j < n/2$.
(iii) α_n maps the point corresponding to $\theta_{k,j}^{-+}$ to a point in C_{k+2j} if $0 \le j < n/2-k$ and in C_{k+2j+1} if $n/2-k \le j < n/2$.
(iv) α_n maps the point corresponding to $\theta_{k,j}^{--}$ to a point in C_{k+2j-1} for $1 \le j \le n/2$.
(v) If k is odd, then n should be added to the suffix of the image corridor C_{2j-k} etc in each of cases (i) to (iv).

We remark that, for fixed even $k \in \{0, \dots, n-1\}$, the images of the points $\theta_{k,j}^{++}$ (for $1 \le j \le n/2-2$) are equally spaced and so they lie in corridors C_{2j-k} that (with the initial and final cases of $C_{-k},\ C_{n-k-2}$) alternate in the sequence $C_{-k}, \dots, C_{-k+n-2}$. Notice that, in $C_0, \dots, C_{n-1}$, the images of our points are in C_q for $q = k-3, k-5, \dots, 3, 1, 0, 2, \dots, k-2, k, \dots, n-k-4$, which means in

$$\{C_0, C_1, \dots, C_{k-3}, C_{k-2}, C_k, C_{k+2}, \dots, C_{n-k-6}, C_{n-k-4}\}. \tag{7.1}$$

However, the equally spaced images of the points $\theta_{k,j}^{+-}$ (for $0 \le j < n/2$) are translated by a distance that depends on k so they can change from C_{2j-k} to C_{2j-k-1} as j passes the value k. Thus these images lie in C_q for

$$q = k-1, k-3, \dots, 3, 1, 0, 2, \dots, k-2, k, k+1, k+3, \dots, n-k-3,$$

which means in

$$\{C_0, C_1, \dots, C_k, C_{k+1}, C_{k+3}, C_{k+5}, \dots, C_{n-k-5}, C_{n-k-3}\}.$$

Notice that there are two more corridors listed here than in (7.1) because j takes two more values in $\theta_{k,j}^{+-}$ than in $\theta_{k,j}^{++}$.

Proof We shall study the effect of multiplying by $(n-1)$ and $(n+1)$. For (i)

$$(n-1)(k(n/2-1)+j)/(n(n-2)) = k/2 + (2j-k)/(2n) + j/(n(n-2))$$

gives a point in C_{2j-k} since, when k is even, $k/2$ is an integer and has no effect. For (ii)

$$\begin{aligned}(n-1)(kn/2+j)/(n^2-2) &= k/2+[(2j-k)n/2+(k-j)]/(n^2-2)\\ &= k/2+[(2j-k-1)n/2+(n/2+k-j)]/(n^2-2)\end{aligned}$$

gives a point in C_{2j-k} if $k-j \geq 0$ and in C_{2j-k-1} if $k-j<0$. For (iii)

$$\begin{aligned}(n+1)(kn/2+j)/(n^2-2) &= k/2+((k+2j)n/2+k+j)/(n^2-2)\\ &= k/2+((k+2j+1)n/2+(k+j-n/2))/(n^2-2)\end{aligned}$$

gives a point in C_{k+2j} if $0 \leq j < n/2-k$ and in C_{k+2j+1} if $n/2-k \leq j < n/2$. For (iv)

$$(n+1)(k(n/2+1)+j)/(n(n+2)) = k/2+((k+2j-1)/(2n)+(n/2+1-j))/(n(n+2))$$

gives a point of C_{k+2j-1}. For (v) we note that if k is odd then $k/2$ contributes $\frac{1}{2}$, which adds n to the corridor number. □

Now we shall study duality for period 2 points in C_k and we will refer to these points by their parameter $\theta^{++}_{k,j}$ etc. It will suffice for us to consider the cases $0 \leq k < n/2$ since the duality in $C_{n/2}, \ldots, C_{n-1}$ is equivalent to duality in these corridors under $(x,x,z) \mapsto (-x,-x,z)$ or $\theta \mapsto \frac{1}{2}-\theta$. First we discuss when the curve joining two dual points can cross a vertical line of fixed or symmetric points because this will guide us on whether a point of $\mathrm{Fix}(\alpha_n^2) \cap \partial\mathcal{P}$ must be dual to another in the *same* corridor.

In this paragraph we assemble information from Proposition 7.1 about various curves crossing one or more fixed or symmetric vertical lines. Among the points of period 2 in $\bigcup_{k=0}^{n-1} C_k$, we have already seen in Proposition 7.1 (iii) and (iv) that, for $0 \leq k < n/2-1$ even,

$$\theta^{++}_{k,k} = k/(2(n-2)) = (kn/2)/(n(n-2))$$

is dual to

$$\theta^{--}_{k-1,n/2-k+1} = k/(2(n+2)) = (kn/2)/(n(n+2))$$

with both points fixed, that $\theta^{++}_{k+1,k+1}$ is dual to $\theta^{--}_{k,n/2-k}$ with both points symmetric, and that the fixed or symmetric curve joining these points crosses the fixed or symmetric (respectively) vertical line at the boundary of C_k. Proposition 7.1 (iii) and (iv) say also that the fixed points $\theta^{--}_{n/2-1,1}$ and $\theta^{--}_{n/2,n/2}$ are dual and the curve of fixed points joining them crosses the fixed vertical line $\theta = 1/4$ (actually at the origin $(x,x,z) = (0,0,0)$). Moreover the symmetric points $\theta^{--}_{n/2-2,2}$ and $\theta^{--}_{n/2+1,n/2-1}$ are dual and the symmetric curve joining them crosses the fixed vertical line $\theta = 1/4$ and the symmetric vertical lines $\theta = 1/4 \pm 1/(2n)$. See Figure 3.

Proposition 7.3. *The curve of period* 2 *points joining* $\theta^{--}_{n/2-2,2}$ *and* $\theta^{--}_{n/2+1,n/2-1}$ *is the only such curve to cross both edges of any corridor.*

Proof If any curve in $\mathrm{Fix}(\alpha_n^2)$ crosses a fixed vertical line at a point r_1 and then an adjacent symmetric vertical line at r_2 then its image reaches from r_1 to $(r_2)\alpha_n$ and, if r_1 does not have $\theta = 1/4$, meets at least a fixed and a symmetric vertical line between these points and then *those* intersections cannot be the image of points between r_1 and r_2. Having $\theta = 1/4$ for r_1 is therefore the only way that the curve joining two points of period 2 can cross two adjacent vertical lines (the edges of some corridor). As there is no other symmetric point in $C^+_{n/2-2} \cup C^-_{n/2-2}$ the curve joining $\theta^{--}_{n/2-2,2}$ and $\theta^{--}_{n/2+1,n/2-1}$ is the only one to cross both edges of any corridor. $\square$

In the situation $n = 12$, Figure 4 shows the unique curve of period 2 points that crosses the two central corridors. This curve crosses the z-axis at approximately $x = \pm 0.3$.

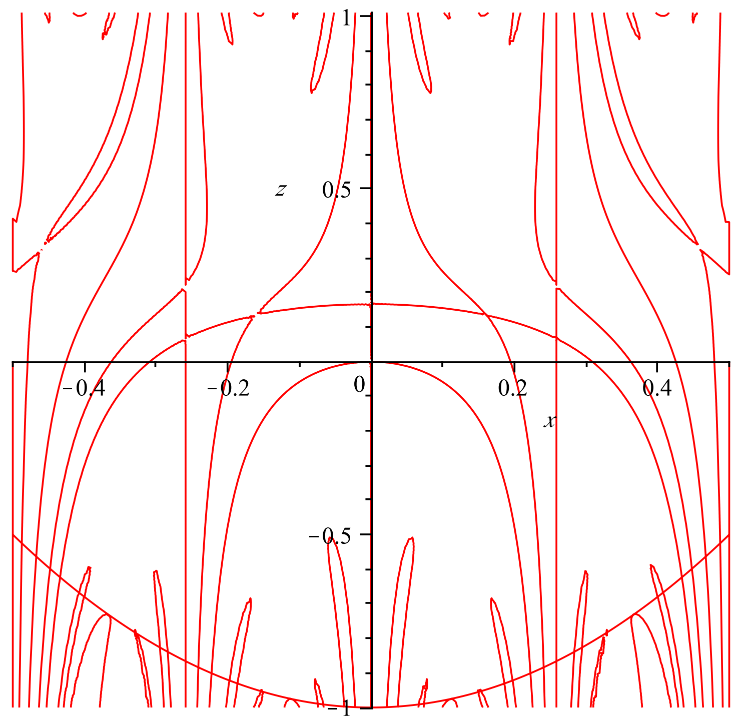

FIGURE 4. Central corridors for α_{12}.

Proposition 7.4. *If $0 < k < n/2$, then the only curves of period 2 points that cross an edge of C_k are:*

(i) the curves of fixed or symmetric points in Proposition 7.1 (iii) and (iv), with the latter including the curve in Proposition 7.3.

(ii) the curve of period 2 points joining the points corresponding to $\theta^{+-}_{k,0}$ *and* $\theta^{-+}_{k-1,n/2-k}$.

Also, the points corresponding to $\theta^{+-}_{k,k}$ *and* $\theta^{-+}_{k,0}$ *are dual.*

Proof If there is a path in $\mathrm{Fix}(\alpha_n^2)$ from $p \in (\mathrm{Fix}(\alpha_n^2) \setminus \mathrm{Fix}(\alpha_n)) \cap (C_k^+ \cup C_k^-)$ that meets the *fixed* vertical edge of C_k first at a point r, say, then α_n takes this part of the curve to a curve from $(p)\alpha_n$ to r that does not meet the edge of any corridor. Assume that k is even. Then $(p)\alpha_n$ is in the corridor C_{k-1} adjacent to C_k (or possibly in C_k itself) and, by Proposition 7.2, this happens precisely with $\theta^{+-}_{k,0}$ mapped to $\theta^{-+}_{k-1,n/2-k}$ or with $\theta^{+-}_{k,k}$ mapped to $\theta^{-+}_{k,0}$. Now $\theta^{-+}_{k-1,n/2-k}$ must be dual to $\theta^{+-}_{k,0}$ by a curve that has non-zero algebraic intersection number with the fixed edge of C_k and is mapped to the reverse of itself. (It could not be dual to $\theta^{+-}_{k,k}$ or $\theta^{-+}_{k,0}$ because the curve joining them would map to a curve joining points in C_k and so having algebraic intersection number 0 with that fixed edge.) Then $\theta^{+-}_{k,k}$ and $\theta^{-+}_{k,0}$ are dual by a curve that is sent to its reverse. From the order of the points along $\partial\mathcal{P}$ the curve in $\mathrm{Fix}(\alpha_n^2)$ joining $\theta^{+-}_{k,k}$ and $\theta^{-+}_{k,0}$ must cross the curve in $\mathrm{Fix}(\alpha_n)$ joining $\theta^{++}_{k,k}$ to $\theta^{--}_{k-1,n/2-k-1}$ (and then there is a path in $\mathrm{Fix}(\alpha_n^2)$ from $\theta^{+-}_{k,k}$ or $\theta^{-+}_{k,0}$ to the fixed edge following part of its curve and then part of that curve of fixed points).

If k is odd a similar discussion shows that $\theta^{+-}_{k,0}, \theta^{-+}_{k-1,n/2-k}$ and $\theta^{+-}_{k,k}, \theta^{-+}_{k,0}$ are dual pairs connected in $\mathrm{Fix}(\alpha_n^2)$ to the *symmetric* edge of C_k and the curve joining each of these pairs is mapped to the curve symmetric to it. □

Figure 1 shows the three curves described here joining the points corresponding to $\theta^{++}_{k,k}$ and $\theta^{--}_{k-1,n/2-k+1}$, $\theta^{+-}_{k,0}$ and $\theta^{-+}_{k-1,n/2-k}$, and $\theta^{+-}_{k,k}$ and $\theta^{-+}_{k,0}$ (for $k = n/2 - 2, n = 12$). They are the two curves that cross the central vertical line and the one that crosses the lower of these two.

Now we can complete the determination of the dual pairing.

Theorem 7.5. *Fix k in $\{0, \dots, n/2-1\}$.*
(i) For $0 < j < k$, $\theta^{++}_{k,j}$ is dual to $\theta^{+-}_{k,j}$. These pairs give $k-1$ stalactites and they are mapped to columns.
(ii) For $k < j \le n/2-2$, $\theta^{++}_{k,j}$ is dual to $\theta^{-+}_{k,j-k}$. These pairs give $n/2-k-2$ columns that map to stalactites.
(iii) For $0 < j < n/2-k$, $\theta^{--}_{k,j}$ is dual to $\theta^{+-}_{k,j+k}$. These pairs give $n/2-k-1$ columns that map to stalagmites.
(iv) For $n/2-k < j \le n/2$, $\theta^{--}_{k,j}$ is dual to $\theta^{-+}_{k,j-1}$. These pairs give k stalagmites and they are mapped to columns.
(v) All the curves joining dual pairs listed in (i)-(iv) are pairwise disjoint and do not meet the curve joining the points corresponding to $\theta^{++}_{k,k}$ and $\theta^{--}_{k-1,n/2-k+1}$ or the one for $\theta^{++}_{k+1,k+1}$ and $\theta^{--}_{k,n/2-k}$. Thus the stalactites are contained in a triangle with one vertical side, one given by the curve for $\theta^{++}_{k,k}$ and $\theta^{--}_{k-1,n/2-k+1}$ and one along $\partial^+\mathcal{P}$ and the stalagmites are contained in a triangle with one vertical side, one given by the curve for $\theta^{++}_{k+1,k+1}$ and $\theta^{--}_{k,n/2-k}$ and one along $\partial^-\mathcal{P}$.
(vi) The points listed in (i)–(iv) account for all the points of $\mathrm{Fix}(\alpha_n^2)$ *in* $C_k^{\pm}$ *except*

$$\theta^{++}_{k,k},\ \theta^{--}_{k,n/2-k},\ \theta^{+-}_{k,k},\ \theta^{+-}_{k,0},\ \theta^{-+}_{k,n/2-k-1},\ \theta^{-+}_{k,0},$$

which we describe next.
(vii) If $0 < k < n/2-1$ *then there are dual pairs* $\theta^{++}_{k,k}, \theta^{--}_{k-1,n/2-k+1}$ *and* $\theta^{+-}_{k,0}, \theta^{-+}_{k-1,n/2-k}$ *whose curves cross from* C_k *to* C_{k-1} *and a pair* $\theta^{+-}_{k,k}$, $\theta^{-+}_{k,0}$ *whose curve lies within* C_k*. These are three columns and they each map to the same column if* k *is even and to the symmetric column if* k *is odd. If* $0 \le k < n/2-2$ *then the penultimate sentence says that the points dual to* $\theta^{--}_{k,n/2-k}$ *and* $\theta^{-+}_{k,n/2-k-1}$ *in* C_k *are* $\theta^{++}_{k+1,k+1}$ *and* $\theta^{+-}_{k+1,0}$ *respectively in* C_{k+1}*. This accounts for the duality of the points in (vi) when* $0 < k < n/2-2$ *.*
(viii) If $k = n/2-2$ *then* $\theta^{-+}_{k,n/2-k-1} = \theta^{-+}_{n/2-2,1}$ *in* $C_{n/2-2}$ *is dual to* $\theta^{+-}_{k+1,0} = \theta^{+-}_{n/2-1,0}$ *in* $C_{n/2-1}$ *and the column joining them is mapped to the symmetric column; but there is no point* $\theta^{++}_{k+1,k+1}$ *and, instead,* $\theta^{--}_{k,n/2-k} = \theta^{--}_{n/2-2,2} = 1/4 - 1/(n+2)$ *is dual to its image* $\theta^{--}_{n/2+1,n/2-1} = 1/4 + 1/(n+2)$ *by a curve of symmetric points that is mapped to itself as in Proposition 7.1 (iv). This accounts for the duality of the points in (vi) when* $k = n/2-2$*.*
(ix) If $k = n/2-1$ *then, as usual,* $\theta^{+-}_{k,k}$ *and* $\theta^{-+}_{k,0}$ *are dual in* $C_{n/2-1}$ *while* $\theta^{+-}_{k,0}$ *and* $\theta^{-+}_{k-1,n/2-k}$ *are dual by a column that crosses from* $C_{n/2-1}$ *to* $C_{n/2-2}$ *and both of these curves are mapped to the symmetric columns. Again there is no point* $\theta^{++}_{k,k}$ *but* $\theta^{--}_{k,n/2-k} = \theta^{--}_{n/2-1,1}$ *is dual to* $\theta^{--}_{n/2,n/2}$ *by a curve through the origin consisting of fixed points, and these are given by* $\theta = 1/4 \pm 1/(2(n+2))$ *as in Proposition 7.1 (iii). This accounts for the duality of the points in (vi) when* $k = n/2-1$*.*
(x) If $k = 0$ *then the points* $\theta^{++}_{k,k}$, $\theta^{+-}_{k,0}$, $\theta^{+-}_{k,k}$, $\theta^{-+}_{k,0}$ *all reduce to the vertex* $(x, x, z) = (1, 1, 1)$ *which does not belong to a curve of fixed points that enters* $\mathcal{P}$*. Together with the two points whose duality is given in (vii) this covers all six points in* C_0 *listed in (vi).*

Proof By Proposition 7.4, the curves in $\mathrm{Fix}(\alpha_n^2)$ through points corresponding to parameters listed in (i) to (iv), and hence also the images of these curves, do not cross the edge of any corridor. Thus a point listed here belongs to the same corridor as its dual and the same is true of their images.

Suppose that k is even. From Proposition 7.2 we recall the image corridors of the pairs of points in (i)–(iv). For $0 < j < k$, $\theta^{++}_{k,j}$ and $\theta^{+-}_{k,j}$ in (i) are both mapped into C_{2j-k}, while, for $k < j \le n/2-2$, $\theta^{++}_{k,j}$ and $\theta^{-+}_{k,j-k}$ in (ii) are both mapped into C_{2j-k}. Again, for $0 < j < n/2-k$, $\theta^{--}_{k,j}$ and $\theta^{+-}_{k,k+j}$ in (iii) are both mapped into C_{k+2j-1}, while, for $n/2-k < j \le n/2$, $\theta^{--}_{k,j}$ and $\theta^{-+}_{k,j-1}$ in (iv) are both mapped into C_{k+2j-1}. Notice that these image corridors are $C_{2-k}, C_{4-k}, \dots, C_{k-4}, C_{k-2}$, not C_k, but then $C_{k+2}, C_{k+4}, \dots, C_{n-k-6}, C_{n-k-4}$ and $C_{k+1}, C_{k+3}, \dots, C_{n-k-5}, C_{n-k-3}$, not C_{n-k-1}, but then $C_{n-k+1}, C_{n-k+3}, \dots, C_{n+k-3}, C_{n+k-1}$. Since $C_{2-k} = C_{k-3}$ and $C_{n+k-1} = C_{n-k}$ etc, these image corridors are all of the corridors except for C_k itself, the symmetric corridor C_{n-k-1} and one adjacent to each of these, namely C_{k-1} and C_{n-k-2}. In particular, they are all distinct which means that the two points mapped into each of them must be dual, proving (i)–(iv). If k is odd then, by Proposition 7.2(v), n is added to the number of the image corridor making it symmetric to the one just described but not affecting the argument.

(v) The curves joining these dual pairs are mapped, according to the proof of (i)–(iv), into different corridors so any two are disjoint because otherwise their image would contain a curve in $\mathrm{Fix}(\alpha_n^2)$ crossing the edge of a corridor, which is excluded by Proposition 7.4. The curves from $\theta^{++}_{k,k}$ and from $\theta^{--}_{k,n/2-k}$ to the edge of C_k map into the corridors C_k and $C_{n/2-k-1}$ respectively and so again are disjoint from the other curves just considered. See Figure 1.

(vi) The set of parameters of points of $\mathrm{Fix}(\alpha_n^2)$ in $C_k^{\pm}$ is given before Proposition 7.2 as

$$\{\theta^{++}_{k,j} : 1 \le j \le n/2-2\} \cup \{\theta^{--}_{k,j} : 1 \le j \le n/2\} \cup \{\theta^{+-}_{k,j}, \theta^{-+}_{k,j} : 0 \le j < n/2\}$$

and these are those listed in (i)–(iv) together with

$$\{\theta^{++}_{k,k},\ \theta^{--}_{k,n/2-k},\ \theta^{+-}_{k,k},\ \theta^{+-}_{k,0},\ \theta^{-+}_{k,n/2-k-1},\ \theta^{-+}_{k,0}\}.$$

(vii) Proposition 7.1 (iii) and (iv) showed that $\theta^{++}_{k,k} = k/(2(n-2))$ is dual to $\theta^{--}_{k-1,n/2-k+1} = k/(2(n+2))$. The other pair was discussed in Proposition 7.4.

(viii) and (ix) The exceptional cases $k = n/2-2$ and $n/2-1$ were included in Proposition 7.1 (iv) and (iii) respectively. See Figure 4.

(x) When $k = 0$, $\theta^{--}_{k,n/2-k} = \theta^{--}_{0,n/2}$ is dual to $\theta^{++}_{1,1}$ and $\theta^{-+}_{k,n/2-k-1} = \theta^{-+}_{0,n/2-1}$ is dual to $\theta^{+-}_{1,0}$ according to the case $k = 1$ of (vii), while $\theta^{++}_{k,k} = \theta^{+-}_{k,k} = \theta^{+-}_{k,0} = \theta^{-+}_{k,0} = 0$ all reduce to $(1,1,1)$. As in [HM] Lemma 2.11, the curve $\gamma^+(x)$ of Proposition 7.1 (iii) of fixed points through the vertex $(1,1,1)$ of $\mathcal{T}$ does not enter $\mathcal{P}$. $\square$

Remarks From (i) of Theorem 7.5, the total number of stalactites is given by $2\sum_{k=1}^{n/2-1}(k-1) = (\frac{n}{2}-1)(\frac{n}{2}-2)$, which agrees with the number $2\sum_{k=0}^{n/2-2}(\frac{n}{2}-k-2)$ of columns in (ii) that are mapped to stalactites. Also, from (iv) there, the total number of stalagmites is $2\sum_{k=0}^{n/2-1} k = \frac{n}{2}(\frac{n}{2}-1)$, which agrees with the number $2\sum_{k=0}^{n/2-1}(\frac{n}{2}-k-1)$ of columns in (iii) that are mapped to stalagmites.

Notice how the fact in (ii) and (iii) that columns join points whose second suffix differs by k is illustrated in Figure 1 as is the fact that the stalactites in (i) and the stalagmites in (iv) are enclosed in the triangles mentioned in (v) bounded by the vertical lines and the curve joining the points corresponding to $\theta^{++}_{k,k}$ and $\theta^{--}_{k-1,n/2-k+1}$ or the one for $\theta^{++}_{k+1,k+1}$ and $\theta^{--}_{k,n/2-k}$. The notation in (i) and (iv) indicates that each stalactite or stalagmite connects adjacent points of $\mathrm{Fix}(\alpha_n^2)\cap\partial\mathcal{P}$ as seen in Figure 1, but actually the order of their endpoints changes at a place that depends on k since, for $0 < j < k$,

$$\frac{k(n/2-1)+j}{n(n-2)} = \theta^{++}_{k,j} < \theta^{+-}_{k,j} = \frac{kn/2+j}{n^2-2} \iff j < \frac{k}{2}\left(1 - \frac{1}{n-1}\right).$$

§8 Some curves of period 2 or 4 not in $\mathcal{P}$

In this section we determine certain types of period 2 and period 4 curves that are not in $\mathcal{P}$. For period 2 we study the situation where the action of α_n on such period two points is given by

$$(x,y,z) \mapsto (y,x,z) \mapsto (x,y,z). \tag{8.1}$$

Recall that

$$\begin{aligned}(x,y,z)\alpha_n &= (v_1, v_2, v_3)\\ &= (-xU_{n-2}(y) + zU_{n-1}(y),\\ &\qquad U_n(x^*)y - U_{n-1}(x^*)(-U_{n-1}(y)x + zU_n(y)),\\ &\qquad yU_{n-1}(x^*) - U_{n-2}(x^*)(-U_{n-1}(y)x + zU_n(y))).\end{aligned} \tag{8.2}$$

Here $x^* = v_1 = -xU_{n-2}(y) + zU_{n-1}(y)$. If, as indicated in (8.1), we require $v_1 = y$, then we have $y = x^* = -xU_{n-2}(y) + zU_{n-1}(y)$, and we can solve for $z = z_n(x, y)$, where

$$z_n(x,y) = \frac{y + xU_{n-2}(y)}{U_{n-1}(y)} = \frac{y + 2xyU_{n-1}(y) - xU_n(y)}{U_{n-1}(y)}.$$

Substituting $z = z_n(x, y)$ into v_2 gives

$$\begin{aligned} v_2 &= U_n(x^*)y - U_{n-1}(x^*)(-U_{n-1}(y)x + zU_n(y)) \\ &= x(U_n(y)^2 + U_{n-1}(y)^2 - 2yU_n(y)U_{n-1}(y)) \\ &= x, \end{aligned}$$

where we get the last equality from Lemma 4.1 (vii).

Now substitute $z = z_n(x, y)$ into v_3 and we get

$$v_3 = \frac{[U_n(y)^2 + U_{n-1}(y)^2 - 2yU_n(y)U_{n-1}(y)](y + 2xyU_{n-1}(y) - xU_n(y))}{U_{n-1}(y)} = z_n(x,y).$$

Here we again use Lemma 4.1 (vii) to give the last equality.

Proposition 8.1. *Fix $n \geq 1$ and $x, y \in \mathbb{R}$. Then we have:*
(i) Let $v = (x, y, z_n(x, y))$. Then $v\alpha_n = (y, x, z_n(x, y))$.
(ii) If $z_n(x, y) = z_n(y, x)$, then $v\alpha_n^2 = v$.

Proof (i) is proved above and (ii) follows from (i), since (i) shows that

$$\begin{aligned} (x, y, z_n(x,y))\alpha_n^2 &= (y, x, z_n(x,y))\alpha_n = (y, x, z_n(y,x))\alpha_n \\ &= (x, y, z_n(y,x)) = (x, y, z_n(x,y)). \quad \square \end{aligned}$$

Define $B_n^{(2)}(x, y)$ to be the numerator of $z_n(x, y) - z_n(y, x)$:

$$\begin{aligned} B_n^{(2)} &= (y + 2xyU_{n-1}(y) - xU_n(y))U_{n-1}(x) - (x + 2xyU_{n-1}(x) - yU_n(x))U_{n-1}(y) \\ &= yU_{n-1}(x) - xU_n(y)U_{n-1}(x) - xU_{n-1}(y) + yU_n(x)U_{n-1}(y). \end{aligned}$$

Next we study curves of period four where the action of α_n looks like

$$(x, y, z) \mapsto (-y, -x, z) \mapsto (-x, -y, z) \mapsto (y, x, z) \mapsto (x, y, z). \tag{8.3}$$

We again use (8.2). If this time we require v_1 to be $-y$, then $x^* = -y = -xU_{n-2}(y) + zU_{n-1}(y)$, and we can solve for $z = Z_n(x, y)$, where

$$Z_n(x,y) = \frac{-y + xU_{n-2}(y)}{U_{n-1}(y)} = \frac{-y + 2xyU_{n-1}(y) - xU_n(y)}{U_{n-1}(y)}.$$

Substituting $z = Z_n(x, y)$ into v_2 gives

$$\begin{aligned} v_2 &= U_n(x^*)y - U_{n-1}(x^*)(-U_{n-1}(y)x + zU_n(y)) \\ &= -x(U_n(y)^2 + U_{n-1}(y)^2 - 2yU_n(y)U_{n-1}(y)) \\ &= -x, \end{aligned}$$

where we get the last equality from Lemma 4.1 (vii).

Similarly, substituting $z = Z_n(x, y)$ into v_3 one finds that $v_3 = z = Z_n(x, y)$. Thus $(x, y, z)\alpha_n = (-y, -x, z)$ if x, y, z satisfy $-xU_{n-2}(y) + zU_{n-1}(y) = -y$.

Repeating the above we start with the expression $(-y,-x,z)$ and apply α_n:

$$\begin{aligned}(-y,-x,z)\alpha_n &= (v_1,v_2,v_3)\\ &= (yU_{n-2}(x)-zU_{n-1}(x),\\ &\quad U_n(x^*)(-x)-U_{n-1}(x^*)(-U_{n-1}(x)y+zU_n(x)),\\ &\quad -xU_{n-1}(x^*)-U_{n-2}(x^*)(-U_{n-1}(x)y+zU_n(x))).\end{aligned}$$

Here $x^* = v_1 = yU_{n-2}(x) - zU_{n-1}(x)$. If we put $v_1 = -x$, as dictated by (8.3), then as in the above we find that $(v_1, v_2, v_3) = (-x, -y, z)$.

Repeating the above two more times (with no additional hypotheses) gives:

Theorem 8.2. *Let n be even and let $x, y, z \in \mathbb{R}$ satisfy*

$$-xU_{n-2}(y) + zU_{n-1}(y) + y = 0 \text{ and } yU_{n-2}(x) - zU_{n-1}(x) + x = 0.$$

Then (x, y, z) is a point of period 4 for α_n with the action given by (8.3). □

Now solving the first equation in the above result for z, and substituting into the second gives a function whose numerator is

$$B_n^{(4)}(x,y) = xU_{n-1}(y) - yU_{n-2}(x)U_{n-1}(y) - yU_{n-1}(x) + xU_{n-2}(y)U_{n-1}(x).$$

It is easy to see that $B_n^{(4)}(x,y)$ is the numerator of $Z_n(x,y) - Z_n(-y,-x)$, similar to the period two case. In fact $B_n^{(4)}(-x,y) = B_n^{(2)}(x,y)$. We draw both curves in Figure 5 for $n = 12$ where the period 2 curve crosses the diagonal $x = y$ in the first rectangle of the first quadrant.

Since $B_n^{(4)}(-x,y) = B_n^{(2)}(x,y)$ the duality for points of period 4 given by (8.3) will then follow from Theorem 8.13 which gives a description of the duality for points of period 2 as determined by the rule (8.1). In order to study duality for such period 2 points we first find where these curves of period 2 meet $\partial\mathcal{T}$.

Lemma 8.3. *For all $n \geq 1$ and all $x, y \in \mathbb{R}$ we have*

$$\begin{aligned}E(x,y,z_n(x,y)) - 1 &= \frac{\left(x - \frac{U_{n+1}(y)}{2} + \frac{U_{n-1}(y)}{2}\right)\left(x - \frac{U_{n-1}(y)}{2} + \frac{U_{n-3}(y)}{2}\right)}{U_{n-1}(y)^2}\\ &= \frac{(x - T_{n+1}(y))(x - T_{n-1}(y))}{U_{n-1}(y)^2}.\end{aligned}$$

Proof Substituting for $z_n(x,y)$ shows that the two sides of the first equality are equal. The second equality follows from the fact that $T_n = \frac{U_n - U_{n-2}}{2}$. □

Now the points of interest are on the curve $E(x,y,z_n(x,y)) = 1$ and also satisfy $B_n^{(2)}(x,y) = 0$. From Lemma 8.3 we see that $E(x,y,z_n(x,y)) = 1$ determines two cases:
(i) $x = T_{n+1}(y)$; and (ii) $x = T_{n-1}(y)$.

Thus to determine the points where these curves of period 2 meet $\partial\mathcal{T}$ we solve $B_n^{(2)}(T_{n\pm1}(y), y) = 0$.

Proposition 8.4. *(i) For all $n, m \geq 1$ we have:*

$$U_n(T_m(x)) = 2\sum_{k=0}^{n/2} T_{(n-2k)m}(x).$$

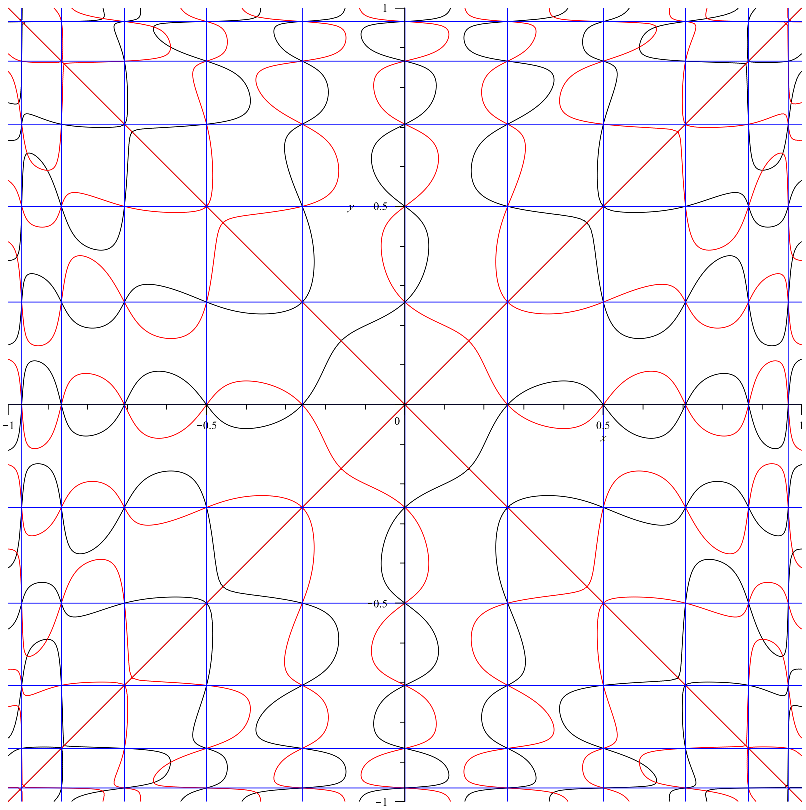

FIGURE 5. Diagonals, fundamental rectangles and curves of period 2 and 4 points for α_{12} projected onto the xy-plane.

(ii) For all $n \geq 1$ we have:

$$B_n^{(2)}(T_{n+1}(y), y) = \frac{1}{2}(U_{(n+1)^2-3}(y) - U_{n^2-2}(y) - U_{2n}(y) + 1).$$

The roots of $B_n^{(2)}(T_{n+1}(y), y)$ are $y = \cos 2\pi\theta$, where

$$\theta = \frac{k}{n^2+2n}, \qquad \theta = \frac{k}{n^2-2}, \qquad \theta = \frac{k}{2n},$$

for any $k \in \mathbb{Z}$.

(iii) $B_n^{(2)}(T_{n-1}(y), y) = \frac{1}{2}(U_{n^2-2}(y) - U_{n^2-2n-2}(y) - U_{2n-2}(y) - 1)$. *The roots of* $B_n^{(2)}(T_{n-1}(y), y)$ *are* $y = \cos 2\pi\theta$, *where*

$$\theta = \frac{k}{n^2 - 2n}, \qquad \theta = \frac{k}{n^2 - 2}, \qquad \theta = \frac{k}{2n},$$

for any $k \in \mathbb{Z}$.

Proof One proves (i), and then for each of (ii), (iii) one proves the first statement using (i), and then uses this to find the roots. The details are left to the reader. □

Proposition 8.5. *The places where the period two points* $(x, y, z_n(x, y))$ *meet* $\partial\mathcal{T}$ *are when*

(i) $x = \cos 2\pi(n+1)\theta, y = \cos 2\pi\theta$, *with* θ *as given in Proposition 8.4 (ii); and*

(ii) $x = \cos 2\pi(n-1)\theta, y = \cos 2\pi\theta$, *with* θ *as given in Proposition 8.4 (iii).* □

Now the denominator of $z_n(x, y) - z_n(y, x)$ is $U_{n-1}(x)U_{n-1}(y)$ and $U_{n-1}(x)$ has roots $x = \cos \frac{\pi j}{n}, j \in \mathbb{Z} \setminus n\mathbb{Z}$. We will thus split the square $[-1, 1]^2$ into rectangles bounded by the lines $x = \cos \frac{\pi j}{n}, y = \cos \frac{\pi k}{n}$; these (closed) rectangles we will call *fundamental rectangles*. Each such fundamental rectangle has four *corners*. Some of the fundamental rectangles meet the boundary of $[-1, 1]^2$; we will call these *boundary rectangles*. A corner $c = (x, y)$ will be called a *diagonal corner* if $x = \pm y$.

Let Δ^+ denote the diagonal $x = y$, let Δ^- denote the diagonal $x = -y$ and let $\Delta^\pm = \Delta^+ \cup \Delta^-$. A *diagonal rectangle* is a rectangle (square) one of whose diagonals is in $\Delta^\pm$.

We split the square $[-1, 1]^2$ into four *triangles* determined by the two diagonals in $[-1, 1]^2$. These are naturally called the *bottom, left, top* and *right* triangles.

Let β_n be the set of points $(x, y, z_n(x, y)) \in \mathbb{R}^3$ having period 2 for α_n, where $B_n^{(2)}(x, y) = 0$. Let

$$\pi_{xy} : \mathbb{R}^3 \to \mathbb{R}^2, \quad (x, y, z) \mapsto (x, y),$$

be the projection onto the xy-plane. If R is a fundamental rectangle, then we will denote the set $\pi_{xy}^{-1}(R) \cap \beta_n$ by $\beta_{n,R}$. Then $\pi_{xy}(\beta_{n,R})$ is a curve in R.

Proposition 8.6. *(i) Each non-boundary, non-diagonal corner of each fundamental rectangle is a point of* β_n.

(ii) If R *is a non-boundary fundamental rectangle, then the only places where* β_n *meets* ∂R *are at the corners of* R.

(iii) Suppose that a part of the curve β_n *is in a fundamental rectangle* R *and exits* R *at a (non-boundary, non-diagonal) corner* c *of* R. *Assume that* $c = (x_0, y_0)$. *Then the rectangle that the curve* β_n *enters (after passing through* c*) is the rectangle opposite* R *relative to* c.

Proof (i) Such a corner of a fundamental rectangle has the form $(x_0, y_0) = (\cos \frac{\pi j}{n}, \cos \frac{\pi k}{n})$, $j, k \in \{1, \ldots, n-1\}$, so that $U_{n-1}(x_0) = U_{n-1}(y_0) = 0$. Thus

$$B_n^{(2)}(x_0, y_0) = y_0 U_{n-1}(x_0) - x_0 U_n(y_0) U_{n-1}(x_0) - x_0 U_{n-1}(y_0) + y_0 U_n(x_0) U_{n-1}(y_0) = 0.$$

(ii) Suppose that $(x_0, y_0) \in \partial R$ where $B_n^{(2)}(x_0, y_0) = 0$. Since $B_n^{(2)}(x, y) = -B_n^{(2)}(y, x)$ we can assume that $y_0 = \cos \frac{\pi k}{n}, k \in \{1, \ldots, n-1\}$, and so

$$\begin{aligned} B_n^{(2)}(x_0, y_0) &= y_0 U_{n-1}(x_0) - x_0 U_n(y_0) U_{n-1}(x_0) - x_0 U_{n-1}(y_0) + y_0 U_n(x_0) U_{n-1}(y_0) \\ &= U_{n-1}(x_0) \cdot (y_0 - x_0 U_n(y_0)). \end{aligned}$$

Now let $\theta = \frac{\pi k}{n}, k \in \{1, \dots, n-1\}$, so that $y_0 = \cos\theta$. Then

$$U_n(y_0) = \frac{\sin(n+1)\theta}{\sin\theta} = \frac{\sin n\theta\cos\theta + \cos n\theta\sin\theta}{\sin\theta} = \cos n\theta = \cos(k\pi).$$

Thus we have $U_n(y_0) = \pm 1$, and if we have

$$0 = B_n^{(2)}(x_0, y_0) = U_{n-1}(x_0) \cdot (y_0 - x_0 U_n(y_0)),$$

then we either have (a) $U_{n-1}(x_0) = 0$, or (b) $y_0 = x_0$, or (c) $y_0 = -x_0$. In each case we see that $x_0 = \cos\frac{\pi h}{n}, h \in \{1, \dots, n-1\}$, as required.

(iii) This result will follow if we can show that the tangent to β_n at the corner c is neither horizontal nor vertical. The relevant slope is obtained by differentiating $B_n^{(2)}(x, y)$ w.r.t. x and solving for $y' = \frac{d}{dx}y(x)$ at the point $c = (x_0, y_0)$. Hence:

$$y'\left[U_{n-1}(x) - xU_{n-1}(x)\frac{d}{dy}U_n(y) - x\frac{d}{dy}U_{n-1}(y) + U_n(x)U_{n-1}(y) + yU_n(x)\frac{d}{dy}U_{n-1}(y)\right]$$

$$= -y\frac{d}{dx}U_{n-1}(x) + U_n(y)U_{n-1}(x) + xU_n(y)\frac{d}{dx}U_{n-1}(x) + U_{n-1}(y) - yU_{n-1}(y)\frac{d}{dx}U_n(x).$$

Using the fact that $U_{n-1}(\cos\frac{\pi j}{n}) = 0$ and the expression for the derivative of $U_{n-1}(x)$ from Lemma 4.2 we obtain:

$$y' = \frac{U_n(x_0) \cdot (U_n(y_0)x_0 - y_0) \cdot (y_0^2 - 1)}{U_n(y_0) \cdot (U_n(x_0)y_0 - x_0) \cdot (x_0^2 - 1)}.$$

Since $x_0 = \cos\frac{\pi j}{n}, y_0 = \cos\frac{\pi k}{n}$ one has $U_n(x_0), U_n(y_0) \in \{\pm 1\}$ and $T_n(x_0) = U_n(x_0), T_n(y_0) = U_n(y_0)$. Since the corner is a non-diagonal corner that is not on the boundary it follows that the numerator and the denominator of the above expression cannot be zero. The result follows. $\square$

As can be seen from Figure 5 (where $n = 12$) it is possible that β_n exits a boundary rectangle at a non-corner point.

Let R be a fundamental rectangle. From Proposition 8.5 the points (x, y) of $\beta_{n,R} \cap \partial\mathcal{T}$ have the form

$$(i) \quad x = \cos\frac{2(n+1)m\pi}{n^2+2n}, \quad y = \cos\frac{2m\pi}{n^2+2n}, \quad 0 \le m \le \frac{n^2+2n}{2};$$

$$(ii) \quad x = \cos\frac{2(n-1)m\pi}{n^2-2n}, \quad y = \cos\frac{2m\pi}{n^2-2n}, \quad 0 \le m \le \frac{n^2-2n}{2};$$

$$(iii) \quad x = \cos\frac{2(n+1)m\pi}{n^2-2}, \quad y = \cos\frac{2m\pi}{n^2-2}, \quad 0 \le m \le \frac{n^2-2}{2};$$

$$(iv) \quad x = \cos\frac{2(n-1)m\pi}{n^2-2}, \quad y = \cos\frac{2m\pi}{n^2-2}, \quad 0 \le m \le \frac{n^2-2}{2}.$$

In Figure 6 we show the points of types (i)-(iv), drawn as crosses, diamonds, circles and asterisks, respectively, for $n = 12$. Compare Figure 6 with Figure 5.

We show that each non-diagonal rectangle R has exactly two points of $\beta_{n,R} \cap \partial\mathcal{T}$; these are either of types (i) and (iv), or of types (ii) and (iii). These will then be dual.

We introduce notation for the fundamental rectangles: note that the square $[-1, 1]^2$ is a union of n^2 fundamental rectangles. Each such rectangle R has a lower left corner with coordinates$(\cos\frac{p\pi}{n}, \cos\frac{q\pi}{n})$. We will call R the (p, q)-*rectangle*. Thus

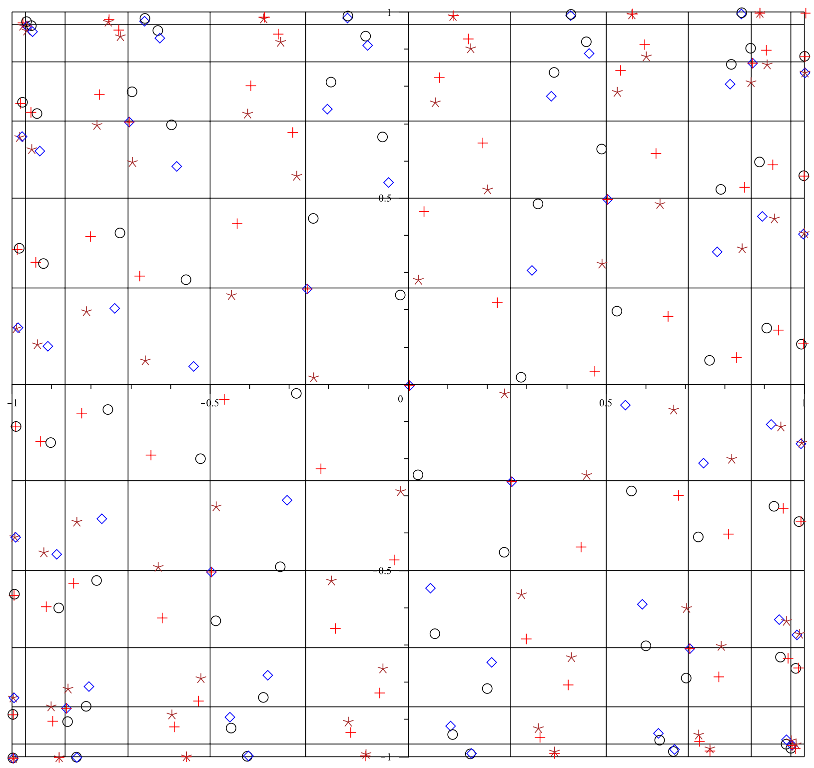

FIGURE 6. Points of type (i)-(iv) for α_{12}.

the $(1,1)$ rectangle is the top right rectangle of $[-1,1]^2$ and the (n,n)-rectangle is the bottom left rectangle of $[-1,1]^2$. We also note that the (p,q)-rectangle is also the $(\pm p+2un, \pm q+2vn)$-rectangle for any $u,v\in\mathbb{Z}$. Let R^o be the interior of R.

Let P_α denote the set of points of type $\alpha, \alpha \in \{(i),(ii),(iii),(iv)\}$.

Lemma 8.7. *Let R be the (p,q)-rectangle.*

(i) Suppose that $(x,y)\in R^o\cap P_{(i)}$ where $x=\cos\frac{2(n+1)k\pi}{n^2+2n}, y=\cos\frac{2k\pi}{n^2+2n}$. Then $q=\lceil\frac{2k}{n+2}\rceil$ and $p=\lceil\frac{2(n+1)k}{(n+2)}\rceil$. No two points of $P_{(i)}$ can be in $R^o\cap P_{(i)}$.

(ii) Suppose that $(x,y) \in R \cap P_{(ii)}$ where $x = \cos\frac{2(n-1)k\pi}{n^2-2n}, y = \cos\frac{2k\pi}{n^2-2n}$. Then $q = \lceil \frac{2k}{n-2} \rceil$ and $p = \lceil \frac{2(n-1)k}{(n-2)} \rceil$. No two points of $P_{(ii)}$ can be in $R^o \cap P_{(ii)}$.

(iii) Suppose that $(x,y) \in R \cap P_{(iii)}$ where $x = \cos\frac{2(n+1)k\pi}{n^2-2}, y = \cos\frac{2k\pi}{n^2-2}$. Then $q = \lceil \frac{2k}{n-2/n} \rceil$ and $p = \lceil \frac{2(n+1)k}{(n-2/n)} \rceil$. No two points of $P_{(iii)}$ can be in R^o.

(iv) Suppose that $(x,y) \in R \cap P_{(iv)}$ where $x = \cos\frac{2(n-1)k\pi}{n^2-2}, y = \cos\frac{2k\pi}{n^2-2}$. Then $q = \lceil \frac{2k}{n-2/n} \rceil$ and $p = \lceil \frac{2(n-1)k}{(n-2/n)} \rceil$. No two points of $P_{(iv)}$ can be in R^o.

Proof (i) We have

$$\cos\frac{q\pi}{n} < \cos\frac{2k\pi}{n^2+2n} < \cos\frac{(q-1)\pi}{n}.$$

This gives

$$\frac{(q-1)\pi}{n} < \frac{2k\pi}{n^2+2n} < \frac{q\pi}{n} \text{ and so } q-1 < \frac{2k}{n+2} < q,$$

which gives $q = \lceil \frac{2k}{n+2} \rceil$. One similarly shows that $p = \lceil \frac{2(n+1)k}{(n+2)} \rceil$.

Now if $(x_j, y_j) \in R^o \cap P_{(i)}, j = 1,2$, where $x_j = \cos\frac{2(n+1)k_j\pi}{n^2+2n}, y_j = \cos\frac{2k_j\pi}{n^2+2n}$, then we would have $\lceil \frac{2(n+1)k_1}{(n+2)} \rceil = \pm \lceil \frac{2(n+1)k_2}{(n+2)} \rceil + 2un$ and $\lceil \frac{2k_1}{(n+2)} \rceil = \pm \lceil \frac{2k_2}{(n+2)} \rceil + 2vn$ for some $u, v \in \mathbb{Z}$. This proves (i).

Cases (ii), (iii), (iv) are similar. □

Now note that if $k = r(n+2)/2, r \in \mathbb{Z}$, then

$$y = \cos\frac{2k\pi}{n^2+2n} = \cos\frac{r\pi}{n}, x = \cos\frac{2(n+1)k\pi}{n^2+2n} = (-1)^r \cos\frac{r\pi}{n} = (-1)^r y.$$

Thus this point is a diagonal corner point. Similarly $x = \cos\frac{2(n+1)k\pi}{n^2+2n} = (-1)^r y$ when $k = rn/2$.

Number the corners of Δ^+ as $0,1,2,\dots$ starting at the level $y = 1$. Number the corners of Δ^- as $0,1,2,\dots$ starting at the level $y = 1$. Let $(i)_k$ denote the point of type (i) with parameter k. This gives part of

Proposition 8.8. *(i) Type (i) points (x,y) $x = \cos\frac{2(n+1)k\pi}{n^2+2n}, y = \cos\frac{2k\pi}{n^2+2n}$ are in $\Delta^\pm$ if and only if $k = \frac{rn}{2}$ or $k = \frac{r(n+2)}{2}$. The even numbered corners of Δ^+ are points of type (i), as are the odd numbered points of Δ^-. These corners of type (i) are where $k = r(n+2)/2$.*

(ii) Type (ii) points (x,y) $x = \cos\frac{2(n-1)k\pi}{n^2-2n}, y = \cos\frac{2k\pi}{n^2-2n}$ are in $\Delta^\pm$ if and only if $k = \frac{rn}{2}$ or $k = \frac{r(n-2)}{2}$. When $k = \frac{r(n-2)}{2}$ the type (ii) point is a corner diagonal point and is also a point of type (i) corresponding to $k = \frac{r(n+2)}{2}$.

(iii) There are no type (iii), (iv) points on the diagonals.

(iv) For $0 < r < n/2$ the points

$$(i)_{r(n+2)/2},\ (ii)_{rn/2},\ (iii)_{rn/2},\ (iv)_{r(n+2)/2}$$

are all in the same diagonal rectangle. This diagonal rectangle meets Δ^+ if and only if r is even.

Proof The proofs are straightforward; for example for (iv) we show that the x-values for $(i)_{r(n+2)/2}$ and $(ii)_{rn/2}$ differ by the correct amount: recalling that the

$(i)_{r(n+2)/2}$ point is a corner and that $0 < r \le n/2 - 1$ we have:

$$\left|\frac{2(n+1)r(n+2)/2}{n^2+2n} - \frac{2(n-1)rn/2}{n^2-2n}\right| = \frac{(n-1)r}{n-2} - \frac{(n+1)r}{n} = \frac{2r}{n(n-2)} \le \frac{2(n/2-1)}{n(n-2)} = \frac{1}{n}. \quad \square$$

This shows that the diagonals have alternately squares with 1 (interior) point (of type (i)) and 3 points, where the common corner point is of type (i) and (ii).

Let ρ denote rotation of $\mathbb{R}^2$ by π about the origin. Let r_+, r_- denote the reflections across Δ^+ and Δ^-, respectively. So $r_- r_+ = \rho$. The group $\langle r_+, r_- \rangle$ acts transitively on the four triangles of $[-1,1]^2$. Thus by the next result we need only consider one of these triangles (the bottom triangle).

Proposition 8.9. *(i)* $\rho(P_\alpha) = P_\alpha$ *for all* $\alpha \in \{(i),(ii),(iii),(iv)\}$.
(ii) $r_+(P_{(i)}) = P_{(i)}$ *and* $r_-(P_{(i)}) = P_{(i)}$.
(iii) $r_+(P_{(ii)}) = P_{(ii)}$ *and* $r_-(P_{(ii)}) = P_{(ii)}$.
(iv) $r_+(P_{(iii)}) = P_{(iv)}, r_+(P_{(iv)}) = P_{(iii)}, r_-(P_{(iii)}) = P_{(iv)}, r_-(P_{(iv)}) = P_{(iii)}$.

Proof (i) If $(x,y) \in P_{(i)}$, then $(x,y) = (\cos\frac{2(n+1)k\pi}{n^2+2n}, \cos\frac{2k\pi}{n^2+2n})$. Then

$$\begin{aligned}
\rho(x,y) = (-x,-y) &= \left(-\cos\frac{2(n+1)k\pi}{n^2+2n}, -\cos\frac{2k\pi}{n^2+2n}\right)\\
&= \left(\cos(\frac{2(n+1)k\pi}{n^2+2n}+\pi), \cos(\frac{2k\pi}{n^2+2n}+\pi)\right)\\
&= \left(\cos\frac{2(n+1)(k+\frac{n^2+2n}{2})\pi}{n^2+2n}, \cos\frac{2(k+\frac{n^2+2n}{2})\pi}{n^2+2n}\right) \in P_{(i)}.
\end{aligned}$$

Here we used the fact that n is even to conclude that $k + \frac{n^2+2n}{2} \in \mathbb{Z}$.

Similarly one shows that $\rho(P_{(ii)})) = P_{(ii)}, \rho(P_{(iii)})) = P_{(iii)}$ and $\rho(P_{(iv)})) = P_{(iv)}$.

(ii) Let $(x,y) = (\cos\frac{2(n+1)k\pi}{n^2+2n}, \cos\frac{2k\pi}{n^2+2n}) \in P_{(i)}$. Then $r_+(x,y) = (y,x)$ and so we need to find some $m \in \mathbb{Z}$ such that

$$(y,x) = \left(\cos\frac{2k\pi}{n^2+2n}, \cos\frac{2(n+1)k\pi}{n^2+2n}\right) = \left(\cos\frac{2(n+1)m\pi}{n^2+2n}, \cos\frac{2m\pi}{n^2+2n}\right) \in P_{(i)}.$$

This is equivalent to solving the rational congruences

$$\frac{2m(n+1)}{n^2+2n} \equiv \frac{2k}{n^2+2n} \mod 2; \text{ and } \frac{2m}{n^2+2n} \equiv \frac{2k(n+1)}{n^2+2n} \mod 2.$$

This is equivalent to solving the integral congruences

$$m(n+1) \equiv k \mod n^2+2n; \text{ and } m \equiv k(n+1) \mod n^2+2n.$$

Since $(n+1)^2 \equiv 1 \mod n^2+2n$ one easily sees that any m solving the first of these equations will automatically solve the other. Now we can solve the first equation since $\gcd(n+1, n^2+2n) = 1$. This does this case and $r_-(P_{(i)}) = P_{(i)}$ follows from the fact that $r_- = \rho r_+$, together with what we have done above.

(iii) The proof of (iii) is similar to the proof of (ii).

(iv) We will prove the first of these as the rest are similar or follow easily. So let $(x,y) \in P_{(iii)}$, so that $(x,y) = \left(\cos\frac{2(n+1)k\pi}{n^2-2}, \cos\frac{2k\pi}{n^2-2}\right)$. Then we need to find $m \in \mathbb{Z}$ such that

$$r_+(x,y) = (y,x) = \left(\cos\frac{2k\pi}{n^2-2}, \cos\frac{2(n+1)k\pi}{n^2-2}\right) = \left(\cos\frac{2(n-1)m\pi}{n^2-2}, \cos\frac{2m\pi}{n^2-2}\right).$$

As in the above this amounts to solving the integral congruences

$$m(n-1) \equiv k \mod n^2-2; \text{ and } m \equiv k(n+1) \mod n^2-2.$$

Now $(n-1)(n+1) \equiv 1 \mod n^2-2$ and (as in the above), we see that solving one of these equations is equivalent to solving both of them. Since $\gcd(n-1, n^2-2) = 1$ we see that there is a solution to the first equation. $\square$

Let $\mathcal{C}_n$ denote the set of corner points of type (i) of diagonal rectangles as described in Proposition 8.8.

Let $\mathfrak{T}_k$ be the part of $[-1,1]^2$ that lies between $y = \cos\frac{k\pi}{n}$, $y = \cos\frac{(k-1)\pi}{n}$ and the diagonals. Thus $\mathfrak{T}_k$ is a trapezoid and from Proposition 8.8 one sees that two of its (opposite) corners are in $\mathcal{C}_n$.

Assume without loss of generality that $\mathfrak{T}_k$ is in the bottom triangle, and that the corners of $\mathfrak{T}_k$ that are in $\mathcal{C}_n$ are the bottom left and the top right corners of $\mathfrak{T}_k$. Note that by the symmetry in x there are an even number of rectangles that are completely contained in $\mathfrak{T}_k$. Denote them by $R_1, \dots, R_{2h}$, ordered from left to right. Here $h = k - n/2 - 1$.

Now the bottom left corner of $\mathfrak{T}_k$ has coordinates $(\cos\frac{k\pi}{n}, \cos\frac{k\pi}{n})$. Since it is a point of $\mathcal{C}_n$ it will also have coordinates $(\cos\frac{(n+1)2m\pi}{n^2+2n}, \cos\frac{2m\pi}{n^2+2n})$, where $m = r(n+2)/2$. Then we have $\frac{k\pi}{n} \equiv \frac{(n+1)2m\pi}{n^2+2n} = \frac{(n+1)\pi r}{n} \mod 2\pi$

Lemma 8.10. *The rectangles $R_1, R_3, \dots, R_{2h-1}$ are the only rectangles R_i to have points of type (i) in them and each such rectangle has exactly one type (i) point.*

Proof The points of type (i) $(x = \cos\frac{2(n+1)k'\pi}{n^2+2n}, y = \cos\frac{2k'\pi}{n^2+2n})$, that are in $\mathfrak{T}_k$ are $(x = \cos\frac{2(n+1)(n/2-i)\pi}{n^2+2n}, y = \cos\frac{2(n/2-i)\pi}{n^2+2n}), k' \in \mathbb{Z}, i \geq 0$. Thus they have the form

$$(x_i = \cos\left(\frac{k\pi}{n} - \frac{2(n+1)i\pi}{n^2+2n}\right), y_i = \cos\left(\frac{k\pi}{n} - \frac{2i\pi}{n^2+2n}\right)).$$

Since $n \geq 4$ we get

$$\frac{(k-2)\pi}{n} < \frac{k\pi}{n} - \frac{2(n+1)\pi}{n^2+2n} < \frac{(k-1)\pi}{n},$$

and so we see that $(x_1, y_1) \in R_1$. If $n = 4$, then this is all we need to show.

If $n > 4$, then since

$$\frac{(k-4)\pi}{n} < \frac{k\pi}{n} - \frac{4(n+1)\pi}{n^2+2n} < \frac{(k-3)\pi}{n},$$

we see that $(x_2, y_2) \in R_3$.

Similarly, if $n > 6$, then since

$$\frac{(k-6)\pi}{n} < \frac{k\pi}{n} - \frac{6(n+1)\pi}{n^2+2n} < \frac{(k-5)\pi}{n},$$

we see that $(x_3, y_3) \in R_5$. Continuing inductively shows that $R_1, R_3, \dots, R_{2h-1}$ each have at least one of these points in them.

Now we count the number of such rectangles in the bottom triangle that have such a point in them. This is

$$(\frac{n}{2} - 1) + (\frac{n}{2} - 2) + \cdots + 2 + 1 = (\frac{n}{2} - 1)(\frac{n}{4}).$$

There are four such triangles, containing a total of $\frac{n(n-2)}{2}$ such points.

From Proposition 8.8 we see that there are $2n+1$ type (i) points on the diagonals, giving a total of $(n^2+2n)/2+1$ such points. This is exactly the number of points of type (i). Lemma 8.10 follows. $\square$

We immediately obtain:

Proposition 8.11. *(a) The rectangles in the bottom triangle that contain points of type (i) (in their interiors) are the (p,q)-rectangles where p is odd and $n-q+1 < p < q$.*

The points of type (i) in the bottom triangle that are on Δ^+ include the top right corners of the (p,p)-rectangles where $n/2 < p \leq n$ is odd; each such rectangle contains one other point of type (i) in its interior. This accounts for the points of type (i) that are on Δ^+.

In the bottom triangle the top left corners of the $(p, n+1-p)$-rectangles where $0 < p < n/2$ is odd, are points of type (i) and these rectangles also contain a point of type (i) in their interior. This accounts for the points of type (i) on Δ^-. $\square$

In a similar way we prove:

Proposition 8.12. *(b) The rectangles inside the bottom triangle that contain points of type (ii) (in their interiors) are the (p,q)-rectangles where $n-q+2 < p < q-1$ and p is even.*

(c) The rectangles inside the bottom triangle that contain points of type (iii) (in their interiors) are the (p,q)-rectangles where $n-q+1 < p < q$ and p is even.

(d) The rectangles inside the bottom triangle that contain points of type (iv) (in their interiors) are the (p,q)-rectangles where $n-q+1 < p < q$ and p is odd.

There are no points of type (iii) or (iv) on $\Delta^\pm$.

The diagonal rectangles have (alternately) either no type (iii) or (iv) points in them or one type (iii) point and one type (iv) point in them. The $(n/2, n/2+1)$-rectangle contains a type (iii) and a (iv) point. $\square$

Thus the following completely describes the duality for points in the bottom triangle, and so by symmetry for all points. Compare Figures 5 and 6. Also, duality for points on $\Delta^\pm$ was done before §8.

Theorem 8.13. *(Bi) Each point of type (i) in the bottom triangle is dual to the unique point of type (iv) that is in the same rectangle. These rectangles are the (p,q)-rectangles in the bottom triangle where p is odd. Every such rectangle (inside the bottom triangle) has one type (i) point and one type (iv) point. This gives the duality for any type (i) or (iv) point in the bottom rectangle.*

(Bii) Each point of type (ii) in the bottom triangle is dual to the unique point of type (iii) that is in the same rectangle. These rectangles are the (p,q)-rectangles in the bottom triangle where p is even. Every such rectangle has one type (ii) point and one type (iii) point. These are dual points if the points are in the interior of the rectangle.

If one of these points is not in the interior of the rectangle, then the top right [or top left] corner is on Δ^- [or Δ^+] and is of type (ii). The type (iii) point in the interior of this rectangle is then dual to the point of type (iv) which is in the reflection of this rectangle across Δ^- [or Δ^+]. This gives the duality for any type (ii) or (iii) point in the bottom rectangle.

The squares above a rectangle having a type (ii) point in their top right [or top left] corner have a point of type (iii) in them. These squares are invariant under

r_- [or r_+] and the image of the type (iii) point is a type (iv) point; these are dual points. □

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